Calculus 1 – Spring 2019 Section 2
Midterm 1–Solutions

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Instructions

- This exam consists of three parts and one extra-credit fourth part. The first three parts involve five questions each and the fourth extra-credit part involves three questions. The parts are organized by the type of answer that is expected. For the three parts, each question weighs exactly seven points, for a total of 105 (out of 100) points.

- In your blue notebook, in the first page, clearly make the following table, noting your final answer corresponding to each question in each cell. For the first part, write out the words "true" or "false" in their entirety (not just "T" or "F"). The front page of your notebook should look as follows before you start solving the exam:

Question number	Part 1	Part 2	Part 3
1	FALSE	∞	$\cos(x)$
2	TRUE	0	x
3	FALSE	0	5
4	FALSE	80	$(5x+3)\,2^x$
5	TRUE	10	x

  In the rest of your notebook, try your best to give an explanation or justification for your answer. Partial credit *may* be given if your final answer is incorrect but the rest of your notebook contains some nonetheless valid reasoning. If your answer is correct no further justification will be required to get full points, i.e. your explanation cannot damage you.

- If you wish to solve the extra-credit problems use a separate new notebook for that and attach it together to your first notebook.

- Write your UNI, *without your name,* clearly on *each* blue notebook you use and submit all of them together. The actual exam sheet (i.e., these very pages) are not to be handed in.

- Write clearly and legibly. Points will not be given if the grader cannot read your final answer.

1 Part 1–Questions whose answer is an element of { true, false }

For each of the following statements, respond with "true" or "false".

1. $x \in \varnothing$ for some x. FALSE: The empty set contains no elements, by its very definition.

2. For the sequence $\mathbb{N} \ni n \mapsto \tan(2\pi n) \in \mathbb{R}$, $\lim_{n \to \infty} \tan(2\pi n)$ exists. (Recall $\tan \equiv \frac{\sin}{\cos}$). TRUE: Note that $\sin(2\pi n) = 0$ and $\cos(2\pi n) = 1$, so that $\tan(2\pi n) = \frac{0}{1} = 0$ for all $n \in \mathbb{N}$. So this is just the constant sequence (of constant zero) in disguise.

3. If $f : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$ has no limit at $x_0 \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ then $\lim_{x \to x_0} (g \circ f)(x)$ necessarily does not exist. FALSE: Here is one possible counter-example (there are many): Take g to be the constant function. Then $g \circ f$ is also the constant function (regardless of what f is). Since the constant function has a limit at all points (equal to that constant), the limit of $g \circ f$ exists.

4. If a function $f : \mathbb{R} \to \mathbb{R}$ is discontinuous at some x_0 then its limit at x_0, $\lim_{x \to x_0} f(x)$, either does not exist or diverges to $\pm\infty$. FALSE: Here is one possible counter-example (there are many): Take $f(x) := \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Then $\lim_{x \to 0} f(x) = 0$ yet f is *not* continuous at zero (since $f(0) = 1$ so that $0 = \lim_{x \to 0} f(x) \neq f(0) = 1$.

5. Two sets A and B whose size is infinite are said to be of the same cardinality iff there is a bijection $A \to B$. $\mathbb{Z}$ and $2\mathbb{N} \equiv \{2, 4, 6, 8, \ldots\}$ are of the same cardinality. TRUE: See HW1Q7.

2 Part 2–Questions whose answer is an element of { 0 } ∪ ℕ ∪ { ∞ }

For each of the following expressions or statements, respond with a single non-negative integer, or ∞, i.e. an element of $\{0\} \cup \mathbb{N} \cup \{\infty\}$.

Recall if A is a set $|A|$ denotes its size, i.e. the (possibly infinite) number of elements it contains.

1. $\left| \mathbb{R}^2 \cap \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \right| = \infty$. The first set, $\mathbb{R}^2$ is the whole two-dimensional continuum plane. The second set is a subset of $\mathbb{R}^2$ (so that the intersection is redundant) corresponding to all points on the unit circle (the circle of radius 1 about the origin, the point $(0,0)$). Since the number of points on the circle is infinite, the size of this set is infinite.

2. If $a : \mathbb{N} \to \mathbb{R}$ is given by $a(n) = \left(-\frac{2}{3}\right)^n$ for all $n \in \mathbb{N}$, $\lim a = 0$. We can rewrite this sequence as a product $\left(-\frac{2}{3}\right)^n = (-1)^n \left(\frac{2}{3}\right)^n$. Now, use the squeeze theorem with the two outer sequences $- \left(\frac{2}{3}\right)^n$ and $\left(\frac{2}{3}\right)^n$. Both of them converge to zero (from below and from above respectively). One possible way to see this is through Claim 6.16, number 4 in the lecture notes (the basic limits). In that claim, take $\alpha = 0$ and $p = -1 + \frac{3}{2}$. Then $\frac{n^\alpha}{(1+p)^n} = \frac{1}{\left(1-1+\frac{3}{2}\right)^n} = \left(\frac{2}{3}\right)^n \to 0$.

3. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \begin{cases} \sin\left(\frac{2\pi}{9}x\right) & x \neq 9 \\ 9 & x = 9 \end{cases}$.

   What is $\lim_{x\to 9} f(x) = 0$. For the purpose of calculating the limit, the piecewise definition is irrelevant since all that matters for the limit is what happens *near* the destination $x = 9$, but not precisely *at* $x = 9$. So we are really at the task of calculating $\lim_{x\to 9} \sin\left(\frac{2\pi}{9}x\right)$. Now since sin is continuous we may push the limit through to get $\sin(2\pi) = 0$.

4. What's $\lim_{x\to 1} \frac{x^{80}-1}{x-1} = 80$ (You may find it useful to know that $x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1} \equiv (x-y)\left(y^{n-1} + xy^{n-2} + x^2y^{n-3} + \cdots + x^{n-1}\right)$).
   We have using the hint

$$\frac{x^{80}-1}{x-1} = \sum_{k=0}^{79} x^k 1^{n-k-1}$$
$$= \sum_{k=0}^{79} x^k$$

   Since the function $x \mapsto \sum_{k=0}^{79} x^k$ is comprised of basic arithmetic operations (addition, multiplication, raising to powers), it is continuous so that when we calculate $\lim_{x\to 1} \sum_{k=0}^{79} x^k$ we may push the limit through to get $\sum_{k=0}^{79} 1^k = \sum_{k=0}^{79} 1 = 1 + 1 + \cdots + 1$ (adding together 80 terms, each term equals to 1). The result is 80.

5. What's $\lim_{x\to 0} \frac{\sin(10x)}{x} = 10$ (You may find it useful to know that $\sin(y) \leq y \leq \tan(y)$ for small y when using the squeeze theorem; also recall $\tan \equiv \frac{\sin}{\cos}$ and $\cos(0) = 1$ when massaging the inequalities). We have $\lim_{x\to 0} \frac{\sin(10x)}{x} = \lim_{x\to 0} 10\frac{\sin(10x)}{10x} = 10 \lim_{x\to 0} \frac{\sin(10x)}{10x}$. Now we can define a new function $g(x) := 10x$ and recall that $\mathbb{R} \setminus \{0\} \ni x \mapsto \frac{\sin(x)}{x}$ was called the sinc function which was presented in the lecture notes in Example 6.32. There we saw that $\lim_{y\to 0} \text{sinc}(y) = 1$. In fact that meant that sinc is continuous since we defined it to be equal to 1 at zero. Hence we have $\lim_{x\to 0} \frac{\sin(10x)}{x} = 10 \lim_{x\to 0} \text{sinc}(10x) = 10\text{sinc}(\lim_{x\to 0} 10x) = 10\text{sinc}(0) = 10 \times 1 = 10$.

3 Part 3–Questions whose answer is a function

For each of the following expressions or statements, respond with an expression that *may* involve a variable $x \in \mathbb{R}$, but doesn't *have* to.

1. Evaluate the convergent limit, $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\sin(x + \varepsilon) - \sin(x) \right) = \cos(x)$.
   (It might be useful to recall that $\sin(x) - \sin(y) = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$)
   To see this, let us calculate using the hint

   $$
   \begin{aligned}
   \frac{1}{\varepsilon} \left(\sin(x + \varepsilon) - \sin(x) \right) &= \frac{2}{\varepsilon} \cos\left(\frac{2x + \varepsilon}{2}\right) \sin\left(\frac{\varepsilon}{2}\right) \\
   &= \cos\left(\frac{2x + \varepsilon}{2}\right) \frac{\sin\left(\frac{\varepsilon}{2}\right)}{\frac{\varepsilon}{2}} \\
   &= \cos\left(\frac{2x + \varepsilon}{2}\right) \operatorname{sinc}\left(\frac{\varepsilon}{2}\right)
   \end{aligned}
   $$

   Now we may use the algebraic laws of limits, using the fact that both cos and sinc are continuous, and knowing that $\operatorname{sinc}(0) = 1$ to get that this converges to $\cos(x)$.

2. Evaluate $\lim_{n \to \infty} \left(\lim_{m \to \infty} \frac{xm}{m+n} \right) = x$. The outer limit in n is there just to confuse you, it doesn't actually do anything. We first calculate the inner limit:

   $$
   \begin{aligned}
   \lim_{m \to \infty} \frac{xm}{m+n} &= x \lim_{m \to \infty} \frac{m}{m+n} \\
   &= x
   \end{aligned}
   $$

   The last line follows because, for fixed n, $\frac{m}{m+n} \to 1$ as $m \to \infty$, as was discussed many times. Then the result is just x, which is independent of n, so as far as n is concerned, this is the constant sequence of constant x. When we now take the limit $n \to \infty$ it doesn't do anything and we get back the same constant x.

3. If $g : \mathbb{R} \to \mathbb{R}$ is the constant function of constant 5, what is $g(f(x)) = 5$. As was already noted before, when g is a constant function, $g(f(x))$ is that constant, regardless of what x or f are.

4. $\lim_{y \to x} (5y + 3) \, 2^y = (5x + 3) \, 2^x$. This question just makes sure you know that if the function involved is continuous (as it is) then you can just push the limit through.

5. The following intersection set

   $$
   \left\{ (a,b) \in \mathbb{R}^2 \mid a^2 + b^2 = x^2 \right\} \cap \left\{ (a,b) \in \mathbb{R}^2 \mid a = x \right\}
   $$

   is a singleton containing a pair of numbers, the first of which equals? Equals x. Indeed, $\left\{ (a,b) \in \mathbb{R}^2 \mid a^2 + b^2 = x^2 \right\}$ is the set of points on the

circle of radius x about the origin in $\mathbb{R}^2$. $\left\{ (a,b) \in \mathbb{R}^2 \mid a = x \right\}$ is the *vertical* line on the plane located at x away from the origin. Hence the two meet at precisely one point, which is $(x,0)$ (indeed, $x^2 + 0^2 = x^2$ for the circle). The first part of the tuple $(x,0)$ is just x.

4 Part 4–Extra credit

Recall the definition of a limit of a sequence:

4.1 Definition. The sequence $a : \mathbb{N} \to \mathbb{R}$ converges to some $L \in \mathbb{R}$ iff for any $\varepsilon > 0$ there is some $N_\varepsilon \in \mathbb{N}$ such that if $n \in \mathbb{N}$ obeys $n \geq N_\varepsilon$ then $|a(n) - L| < \varepsilon$.

Find L and N_ε such that given any $\varepsilon > 0$, the following sequence $a : \mathbb{N} \to \mathbb{R}$ fulfills the above criterion for its convergence. Note: for N_ε (but not L) more than one answer is possible.

1. For $a(n) = \frac{1}{\sqrt[3]{n}}$. The limit converges to $L = 0$. To find N_ε, let us make the following manipulations on $|a(n) - L| < \varepsilon$:

$$|a(n) - L| \qquad < \qquad \varepsilon$$
$$\text{Plug in what's } a(n) \text{ and } L$$
$$\left| \frac{1}{\sqrt[3]{n}} \right| \qquad < \qquad \varepsilon$$
$$\text{Use the fact that } \frac{1}{\sqrt[3]{n}} > 0$$
$$\frac{1}{\sqrt[3]{n}} \qquad < \qquad \varepsilon$$
$$\text{Raise to the power } 3$$
$$\frac{1}{n} \qquad < \qquad \varepsilon^3$$
$$\text{Take reciprocal}$$
$$n \qquad > \qquad \frac{1}{\varepsilon^3}$$

So N_ε can be taken as any integer larger than $\frac{1}{\varepsilon^3}$.

Recall the definition of a limit of a function:

4.2 Definition. The function $f : \mathbb{R} \to \mathbb{R}$ converges to some $L \in \mathbb{R}$ at $x_0 \in \mathbb{R}$ iff for any $\varepsilon > 0$ there is some $\delta_\varepsilon > 0$ such that if $x \in \mathbb{R}$ obeys $|x - x_0| < \delta_\varepsilon$ then $|f(x) - L| < \varepsilon$.

Find L and δ_ε such that given any $\varepsilon > 0$, the following functions $\mathbb{R} \to \mathbb{R}$ fulfill the above criterion for their convergence. Note: for δ_ε more than one answer is possible.

2. $f(x) = 2^{-\frac{1}{(x-5)^2}}$ with $x_0 = 5$ (Recall $\log_2$ (the inverse of $x \mapsto 2^x$) is monotone increasing). We have $L = 0$ and δ_ε may be taken as $\delta_\varepsilon =$

$\sqrt{\frac{1}{-\log_2(\varepsilon)}}$. Let us rewrite the goal inequality to see how this works

$\lvert f(x) - L \rvert$	$<$	ε

Plug in f and L

$\left\lvert 2^{-\frac{1}{(x-5)^2}} \right\rvert$	$<$	ε

2^y is always positive

$2^{-\frac{1}{(x-5)^2}}$	$<$	ε

Take $\log_2$ and use monotone increasing property

$-\dfrac{1}{(x-5)^2}$	$<$	$\log_2(\varepsilon)$

Multiply by -1

$\dfrac{1}{(x-5)^2}$	$>$	$-\log_2(\varepsilon)$

Take reciprocal

$(x-5)^2$	$<$	$\dfrac{1}{-\log_2(\varepsilon)}$

Take square root

$\lvert x - 5 \rvert$	$<$	$\sqrt{\dfrac{1}{-\log_2(\varepsilon)}}$

and so we find the result can be $\delta_\varepsilon := \sqrt{\frac{1}{-\log_2(\varepsilon)}}$ (Recall for $\varepsilon < 1$, $\log_2(\varepsilon) < 0$, so the square root makes sense).

3. $f(x) = x^x$ with $x_0 = 0$. (You may use the following facts:
(1) $\lvert x - y \rvert < \varepsilon$ iff $y - \varepsilon < x < y + \varepsilon$; (2) $\log$ is monotone increasing;
(3) $\log(a^b) = b \log(a)$; (4) $\log(x) \geq 0$ for $x \geq 1$ and $\log(x) \leq 0$ for $x \leq 1$;
(5) $\log(x) \geq 2\left(1 - \frac{1}{\sqrt{x}}\right)$ for all x; (6) $\sqrt{x} - 1 \geq -1$).

We have $L = 1$ and $\delta_\varepsilon = \left(-\frac{1}{2}\log(1-\varepsilon)\right)^2$ can be taken as

$\lvert f(x) - 1 \rvert$	$<$	ε
$-\varepsilon <$	$x^x - 1$	$< \varepsilon$
$1 - \varepsilon <$	x^x	$< 1 + \varepsilon$

Take $\log$ of inequalities

$\log(1-\varepsilon) <$	$\log(x^x)$	$< \log(1+\varepsilon)$

Use the fact $\log(a^b) = b\log(a)$

$\log(1-\varepsilon) <$	$x\log(x)$	$< \log(1+\varepsilon)$

Note that for $x < 1$ and $\varepsilon > 0$, $\log(1+\varepsilon)$ is always positive and $\log(x)$ is always negative, so the right inequality is *always* satisfied. For the second inequality, we use the lower bound from the hint $\log(x) \geq 2\left(1 - \frac{1}{\sqrt{x}}\right)$

so that $x \log (x) \geq 2x \left(1 - \frac{1}{\sqrt{x}}\right) = 2\sqrt{x} (\sqrt{x} - 1) \geq -2\sqrt{x}$. The last step follows from the last hint. We hence find that we need x to satisfy only

$$\log (1 - \varepsilon) \quad < \quad -2\sqrt{x}$$

$$\text{Multiply by } -\tfrac{1}{2}$$

$$-\frac{1}{2} \log (1 - \varepsilon) \quad > \quad \sqrt{x}$$

$$\text{Take square}$$

$$\left(-\frac{1}{2} \log (1 - \varepsilon)\right)^2 \quad > \quad |x|$$

So we can take $\delta_\varepsilon := \frac{1}{4} \left(\log (1 - \varepsilon)\right)^2$.