Instructions

• This exam consists of ten questions (some with sub-questions). Each question is worth ten points to a total of one hundred points. Justify your answer as much as your reasonably can: the questions are open. You do not need to simplify your calculations.

• The exam time is from 4:10pm until 7:00pm: two hours and fifty minutes.

• No calculators are allowed.

• Write your UNI without your name clearly on each blue notebook you use and submit all your notebooks bundled together.

• Write clearly and legibly. Points will be taken off if the grader cannot read your answer.

• You may use any analog source material you wish: your notebook, prepared notes, the official lecture notes, or textbooks. You may not use any digital instruments, including and not limited to: smart phones, watches, laptops, tablets.
1 The exam

1.1 Exercise. Find the following limits or show that they do not exist:
1. \[ \lim_{x \to 0} \frac{\sin(x) - x}{x^3} \]
2. \[ \lim_{x \to \infty} \frac{x^2 + x \cos(x)}{\sin(x)-3x^2} \]
3. \[ \lim_{x \to \infty} x (\arctan(3x) - \frac{\pi}{2}) \]

1.2 Exercise. Find the derivative of the following functions:
1. \[ \log \circ \arctan + \arctan \circ \log + \arctan \cdot \log \]
2. \[ x \mapsto \sin(x) \]
3. \[ x \mapsto \sqrt{x + 1} (3x + 6)^2 \]

1.3 Exercise. Find all extrema of the function \( R \ni x \mapsto x^3 e^{-x} \in R \), decide which are global, and which are maxima or minima.

1.4 Exercise. A cone has a side of length \( \ell \). Find the height that maximizes the volume. Recall the volume of a cone is one-third of the height times the area of the base.

1.5 Exercise. Let \( f \) be differentiable with \( f'(x) \geq 0 \) for all \( x \). What can you say about \( \int_{3}^{7} f \) in each of the following cases:
1. \( f(3) = 5 \)
2. \( f(7) = 5 \)
3. \( f(3) = f(7) = 5 \)

1.6 Exercise. Estimate the following using the linear approximation:
1. \( \sqrt[3]{27} \)
2. \( \arcsin(0.6) \)

1.7 Exercise. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is some differentiable function for which we don’t know the explicit formula, but we do know that for all \( x \in \mathbb{R} \),
\[ e^{4x} - (x + 1) e^{f(x)} = \arcsin(x) \]
Note that at \( x = 0 \), this equation yields
\[ 1 - e^{f(0)} = 0 \]
which in turn implies \( f(0) = 0 \). By differentiating both sides of the equation and using the chain rule, find \( f'(0) \).

1.8 Exercise. Suppose that \( f \) is differentiable and injective, and that
\[
\begin{align*}
  f(-2) &= 4 \\
  f(0) &= 1 \\
  f(3) &= -2 \\
\end{align*}
\]
and
\[
\begin{align*}
  f'(-2) &= -7 \\
  f'(0) &= -5 \\
  f'(3) &= -4. \\
\end{align*}
\]
Find \( (f^{-1})'(−2) \).
1.9 Exercise. Initially there are 26 grams of radioactive mousium with a half-life of 20 years and 30 grams of radioactive duckium with a half-life of 15 years. When will there be just as much radioactive mousium as there is radioactive duckium?

Hint: Radioactive decay follows exponential law, that is, the amount of material \( M(t) \) at a given instant of time \( t \) is

\[
M(t) = M(0)e^{-\frac{\log(2)}{T_{0.5}}t} \quad (t \geq 0)
\]

where \( M(0) \) is the amount of material in the beginning of time (at \( t = 0 \)) and \( T_{0.5} \) is the half-life of said material.

1.10 Exercise. Find the following integrals:

1. \( \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan x \, dx \) (hint: use change of variables)

2. \( \int_{-2}^{2} \sin(x + x^3 + x^5) \, dx \) (hint: use the fact that for any \( a > 0 \), \( \int_{-a}^{a} f = 0 \) if \( f(-x) = -f(x) \) for all \( x \geq 0 \)).

3. \( \int_{0}^{2} \sqrt{4-x^2} \, dx \) (hint: first figure out what is the shape of curve of the integrand, and then use known formulas, rather than directly trying to integrate this (though you certainly may)).