Instructions

• This exam consists of one part and one extra-credit part. Justify your answer as much as your reasonably can: the questions are open. Remember in the first part no proofs are necessary (unless otherwise stated), but you should convince the reader that you gave your answer based on more than a wild guess. You may collect 100/100 points in the first part. The extra credit part is worth 60 points.

• Write your UNI without your name clearly on each blue notebook you use and submit all your notebooks bundled together.

• Write clearly and legibly. Points will be taken off if the grader cannot read your answer.

• You may use any analog source material you wish: your notebook, prepared notes, the official lecture notes, or textbooks. You may not use any digital instruments, including and not limited to: smart phones, watches, laptops, tablets.
1 Ordinary exercises

1.1 Exercise. [10 points] Complete the following definition: “The sequence $a : \mathbb{N} \to \mathbb{R}$ diverges to $+\infty$ iff for any $M > 0$...”

Solution. This is Definition 6.6: A sequence $a : \mathbb{N} \to \mathbb{R}$ is said to go to diverge to $+\infty$ iff for any $M > 0$, there is some number $N \in \mathbb{N}$ (this number may depend on $M$) such that for all $n \in \mathbb{N}$ obeying $n \geq N$, the following condition holds:

$$a(n) \geq M.$$  

1.2 Exercise. [20 points] Find all global and local maximum and minimum of the following function: $f : [-2,1] \to \mathbb{R}$ given by the formula

$$[-2,1] \ni x \mapsto -|x| \in \mathbb{R}.$$  

Solution. The sketch of the graph of the function $f$ looks like this:

![Graph of f](image)

from which it is very easy to read off the global / local extrema: the left boundary point $-2$ is a global minimum, the point of non-differentiability zero is a global maximum, and the right boundary point $1$ is a local minimum.

Of course, we would like to be able to ascertain this even without making a sketch (one big point of finding local and global extrema is to be able to make a sketch) which in this case is quite easy to do, but still. For this reason we follow Corollary 8.51 to find the candidates for the extrema: (1) boundary points of the domain, $-2$ and $1$; (2) points of non-differentiability, $0$; and (3) stationary points (these don’t exist for our particular choice of $f$). The derivative is $f' : [-2,1] \setminus \{0\} \to \mathbb{R}$ given by

$$x \mapsto \begin{cases} +1 & x < 0 \\ -1 & x > 0 \end{cases} \ (x \in [-2,1] \setminus \{0\}).$$  

Hence the function is monotone increasing after the left boundary, which makes it at least a local minimum, and likewise the right boundary is (at least) a local maximum. Since the derivative changes sign from increasing to decreasing at the one point where it is not defined, that point must be (at least) a local maximum.

Now we want to figure which of all these local extrema are global. For this we simply evaluate the function at these points:

$$f(-2) = -2$$

$$f(0) = 0$$

$$f(1) = -1$$

The lowest height is $-2$ which is attained at $-2$, so $-2$ is the global minimum. Thus $0$ is the global maximum and $1$ is where a local minimum is attained.
1.3 Exercise. [30 points] Evaluate
\[ \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 + x^2} \, dx = \, ? \]

Hint: (1) First evaluate the integral on \([-R, R]\), for some fixed \(R > 0\), without taking the limit. (2) Once you find the result of the integral, treat this as an ordinary function of \(R\) for which you can now take the limit \(R \to \infty\).

Solution. We follow the hints:

1. Using Example 9.38 we see that we can employ the change of variable \(\varphi = \tan\) (with domain \((-\frac{\pi}{2}, \frac{\pi}{2})\)); then \(\varphi' = \frac{1}{\cos^2}\). We remark that
\[ \frac{1}{1 + \tan^2} \frac{1}{\cos^2} = 1 \]
so that with \(f(x) := \frac{1}{1 + x^2}\) we get
\[ 1 = (f \circ \varphi)' \]
hence change of variables (Theorem 9.35) implies, with the notation \(\arctan := \varphi^{-1}\) (i.e. the inverse of \(\tan\), which exists, since \(\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}\) is bijective as you may verify, hence \(\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})\) is its inverse, which exists)
\[ \int_{-R}^{R} \frac{1}{1 + x^2} \, dx = \int_{\arctan(-R)}^{\arctan(R)} f(\varphi(u)) \varphi'(u) \, du = \arctan(R) - \arctan(-R) . \]

2. We now try to evaluate the limit
\[ \lim_{R \to \infty} (\arctan(R) - \arctan(-R)) = \, ? . \]
Since \(\lim_{\theta \to \frac{\pi}{2}} \tan(\theta) = \infty\), it makes sense that \(\arctan(\infty) = \frac{\pi}{2}\) and likewise \(\lim_{\theta \to -\frac{\pi}{2}} \tan(\theta) = -\infty\) implies \(\arctan(-\infty) = -\frac{\pi}{2}\). Hence (non-rigorously) we get
\[ \lim_{R \to \infty} (\arctan(R) - \arctan(-R)) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{\pi} . \]

If you write this explanation on the exam, you’ll get full points.
For the sake of completeness, let me show the honest way to show that \(\lim_{R \to \infty} \arctan(R) = \frac{\pi}{2}\). We have the equivalent inequalities
\[ \left| \arctan(R) - \frac{\pi}{2} \right| < \varepsilon \]
\[ \frac{\pi}{2} - \varepsilon < \arctan(R) < \frac{\pi}{2} + \varepsilon . \]
Since \(\arctan\) has values in \((-\frac{\pi}{2}, \frac{\pi}{2})\) (this is its co-domain) the right inequality is always automatically satisfied as long as \(\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})\) is well-defined. Since \(\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}\) is increasing, let us take tan of both sides of the left inequality to get
\[ \tan \left( \frac{\pi}{2} - \varepsilon \right) < R . \]
Hence, we can fulfill the limit definition with threshold \(R > \tan \left( \frac{\pi}{2} - \varepsilon \right)\) for any \(\varepsilon > 0\) (arbitrarily small), using the fact that \(\tan\) is monotone increasing.
A similar argument proves the other limit.
1.4 Exercise. [10 points] Find the linear approximation of the function the function \( \exp : \mathbb{R} \to \mathbb{R} \) near zero, and state how small does the argument have to be for a given level of precision.

Solution. We have using Section 8.4 that, since \( \exp (0) = 1 \) and \( \exp' (0) = 1 \),

\[
\exp (0 + \varepsilon) \approx \exp (0) + \varepsilon \exp' (0) + \ldots
\]

i.e.,

\[
\exp (\varepsilon) \approx 1 + \varepsilon + \ldots
\]

for small \(|\varepsilon|\). If you wrote this it would give you 7 points already. Let us say that one wants precision level \( p > 0 \) (for some small \( p \)), i.e., that

\[
|\exp (\varepsilon) - (1 + \varepsilon)| < p.
\]

We know that \( \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\exp (\varepsilon) - 1) \) exists and equals 1, but to proceed we need the exact way in which this happens, i.e. we need to figure out the \( \varepsilon - \delta \) definition. I.e., let \( a > 0 \) be given. Then we are seeking some \( b_a > 0 \) such that if \(|\varepsilon| < b_a\) then

\[
\left| \frac{1}{\varepsilon} (\exp (\varepsilon) - 1) - 1 \right| < a,
\]

which is equivalent to

\[
1 + \varepsilon - a |\varepsilon| < \exp (\varepsilon) < 1 + \varepsilon + a |\varepsilon|.
\]

As always with limits we try to solve this inequality backwards (for \( \varepsilon \)). So let us take the logarithm of both sides (noting that the logarithm is monotone increasing!) to get

\[
\log (1 + \varepsilon - a) < \varepsilon < \log (1 + \varepsilon + a).
\]

Using Claim 10.3 we have that \( \log (x) \leq x - 1 \) and \( \log (x) \geq 1 - \frac{1}{x} \) for all \( x > 0 \). Hence \( \log (1 + \varepsilon - a) \leq \varepsilon - a \) and \( \log (1 + \varepsilon + a) \geq 1 - \frac{1}{1 + \varepsilon + a} \). Hence this above double inequality would be implied by having the following double inequality hold:

\[
\varepsilon - a < \varepsilon < 1 - \frac{1}{1 + \varepsilon + a}.
\]

But the left one is always true (since \( a > 0 \)), and so we concentrate on what \( \varepsilon \) has to do so that the right one holds. This is apparently equivalent to

\[
\varepsilon < \frac{\varepsilon + a}{1 + \varepsilon + a}
\]

\[
\Downarrow
\]

\[
\varepsilon^2 + a \varepsilon - a < 0
\]

\[
\Downarrow
\]

\[
\frac{1}{2} \left( -a - \sqrt{a^2 + 4a} \right) < \varepsilon < \frac{1}{2} \left( -a + \sqrt{a^2 + 4a} \right)
\]

\[
\Uparrow \text{ (Since } \sqrt{a^2 + 4a} > 0 \text{)}
\]

\[
-\frac{1}{2} \left( -a + \sqrt{a^2 + 4a} \right) < \varepsilon < \frac{1}{2} \left( -a + \sqrt{a^2 + 4a} \right)
\]

Hence we find that if we define

\[
b_a := \frac{1}{2} \left( \sqrt{a^2 + 4a} - a \right)
\]

then if we pick \( \varepsilon \) such that both conditions are true:

1. \(|\varepsilon| < 1 \)
2. $|\varepsilon| < b_p$

Then

$$1 + \varepsilon - p < \exp(\varepsilon) < 1 + \varepsilon + p.$$ 

1.5 Exercise. [30 points] Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} 
\sin(x) & x \geq 0 \\
x & x < 0 
\end{cases} \quad (x \in \mathbb{R}).$$

1. [6 points] Determine where $f$ is continuous.

2. [6 points] Determine where $f$ is differentiable, and where it is, calculate its derivative $f'$.

3. [6 points] Determine where $f'$ is continuous.

4. [6 points] Determine where $f'$ is differentiable, and where it is, calculate its derivative $f''$.

5. [6 points] Determine where $f''$ is continuous.

Solution. We follow the various steps. In general, since both parts of the piecewise definition of $f$ are “nice” functions, we only have to care about what happens at zero: $\sin$ and $x \mapsto x$ are going to be differentiable (and hence continuous, by Theorem 8.12). In conclusion we only have to study continuity / differentiability at zero.

1. $f$ looks like

We check continuity at zero: $f(0) \equiv \sin(0) = 0$ by definition. since the left side of the piecewise definition is also zero as its argument approaches zero, i.e.

$$\lim_{y \to 0^-} y = 0$$

the function is indeed continuous at zero. Hence $f$ is actually continuous everywhere.

2. $f'$ looks like
We have differentiability at least on $\mathbb{R} \setminus \{0\}$, where

$$f'(x) = \begin{cases} 
\cos(x) & x > 0 \\
1 & x < 0
\end{cases}$$

to study whether $f'$ exists at zero we must calculate

$$f'(0) \overset{?}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} f(\varepsilon)$$

this limit may be calculated either from the left or the right, and both have to match if the general limit exists:

$$\lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} f(\varepsilon) = \lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} = 1$$

and also using l'Hopitals’ rule:

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \sin(\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \cos(\varepsilon) = 1$$

Since both one-sided limits exist and match, the general limit exists, and we find $f'$ is actually defined on the whole of $\mathbb{R}$ and equals

$$f'(x) = \begin{cases} 
\cos(x) & x \geq 0 \\
1 & x < 0
\end{cases}$$

3. We have

$$\lim_{x \to 0} \cos(x) = 1$$

so $\lim_{x \to 0} f'(x) = 1 = f'(0)$. Hence $f'$ is continuous everywhere.

4. $f''$ looks like
We have a-priori $f''$ defined on $\mathbb{R} \setminus \{0\}$ as

$$f''(x) = \begin{cases} -\sin(x) & x > 0 \\ 0 & x < 0 \end{cases}$$

We only need to verify whether this really exists at zero. I.e., where $f'$ is differentiable at zero. Let us try to test this:

$$f''(0) \overset{?}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (f'(\varepsilon) - f'(0)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (f'(\varepsilon) - 1).$$

Now we separate into two one-sided limits:

$$\lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} (f'(\varepsilon) - 1) = \lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} (1 - 1) = 0$$

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (f'(\varepsilon) - 1) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (\cos(\varepsilon) - 1) = \lim_{\varepsilon \to 0^+} \frac{-\sin(\varepsilon)}{1} = 0.$$

Since both exist and match, $f'$ is indeed differentiable at zero and its derivative equals zero there, so we really $f''$ defined on the whole of $\mathbb{R}$ and equals

$$f''(x) = \begin{cases} -\sin(x) & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

5. $f''$ is actually also continuous at zero, since $\lim_{\varepsilon \to 0} -\sin(\varepsilon) = 0$.

6. Even though this wasn’t asked, let us proceed for completeness. From the last picture you see that $f''$ is not differentiable at zero, and indeed, $f'''$ looks like
and the reason is

\[
f'''(0) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( f''(\varepsilon) - f''(0) \right)
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} f''(\varepsilon).
\]

Now if we come from the left we get

\[
\lim_{\varepsilon \to 0^-} \frac{1}{\varepsilon} f''(\varepsilon) = \lim_{\varepsilon \to 0^-} \frac{0}{\varepsilon} = 0.
\]

However if we come from the right we get

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} f''(\varepsilon) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (-\sin(\varepsilon))
= -\lim_{\varepsilon \to 0^+} \text{sinc}(\varepsilon)
= -1.
\]

Both limits do exist, but they do not match, so the general limit does not exist, and hence \(f''\) is not differentiable at zero!

## 2 Extra credit

### 2.1 Exercise. [20 points] Give an example of a differentiable function \(f : [a, b] \to \mathbb{R}\) which is convex and \(f'\) is monotone decreasing.

**Solution.** This is impossible, by Claim 8.57!

### 2.2 Exercise. [20 points] Using the product rule of differentiation and the fundamental theorem of calculus, prove the integration by parts theorem: If \(f, g\) are two differentiable functions \([a, b] \to \mathbb{R}\) whose derivatives are integrable, then

\[
\int_a^b fg' = fg|_a^b - \int_a^b f'g.
\]

**Solution.** This is Theorem 9.41. We integrate the product (Leibniz) rule \((fg)' = f'g + fg'\) to get

\[
\int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'.
\]
we use the fundamental theorem of calculus (Theorem 9.32) to simplify the left hand side as

\[ fg|_a^b = \int_a^b f'g + \int_a^b fg' \]

which is the claim, up to re-arrangements.

2.3 Exercise. [20 points] Let \( n \in \mathbb{N} \) be given such that \( n \geq 2 \). Use integration by parts to find a formula connecting

\[ \int_a^b (\cos (x))^n \, dx \]

and

\[ \int_a^b (\cos (x))^{n-2} \, dx \]

Hint: Along the way, it will be useful to use \( \sin^2 + \cos^2 = 1 \).

Solution. We have

\[
\int_a^b \cos^n = \int_a^b \cos^{n-1} \cos \\
= \int_a^b \cos^{n-1} \sin' \\
\quad \text{(Use } (\cos^{n-1})' = (n-1) \cos^{n-2} (-\sin) \text{)} \\
= \cos^{n-1} \sin|_a^b + \int_a^b (n-1) \cos^{n-2} \sin^2 \\
\quad \text{(Use } \sin^2 + \cos^2 = 1 \text{)} \\
= \cos^{n-1} \sin|_a^b + \int_a^b (n-1) \cos^{n-2} (1 - \cos^2) \\
= \cos^{n-1} \sin|_a^b + (n-1) \int_a^b \cos^{n-2} - (n-1) \int_a^b \cos^n \\
\]

so solving this last equation for \( \int_a^b \cos^n \) we get

\[ \int_a^b \cos^n = \frac{1}{n} \cos^{n-1} \sin|_a^b + \frac{n-1}{n} \int_a^b \cos^{n-2} . \]

which is the final answer.