Instructions

• This exam consists of one part and one extra-credit part. Justify your answer as much as your reasonably can: the questions are open. Remember in the first part no proofs are necessary (unless otherwise stated), but you should convince the reader that you gave your answer based on more than a wild guess. You may collect 100/100 points in the first part. The extra credit part is worth 60 points.

• Write your UNI without your name clearly on each blue notebook you use and submit all your notebooks bundled together.

• Write clearly and legibly. Points will be taken off if the grader cannot read your answer.

• You may use any analog source material you wish: your notebook, prepared notes, the official lecture notes, or textbooks. You may not use any digital instruments, including and not limited to: smart phones, watches, laptops, tablets.
1 Ordinary exercises

1.1 Exercise. [10 points] Complete the following definition: “The sequence $a : \mathbb{N} \to \mathbb{R}$ diverges to $+\infty$ iff for any $M > 0$..." 

1.2 Exercise. [20 points] Find all global and local maximum and minimum of the following function: $f : [-2, 1] \to \mathbb{R}$ given by the formula 

$$ [-2, 1] \ni x \mapsto -|x| \in \mathbb{R}.$$ 

1.3 Exercise. [30 points] Evaluate 

$$ \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1+x^2} \, dx = ? $$  

Hint: (1) First evaluate the integral on $[-R, R]$, for some fixed $R > 0$, without taking the limit. (2) Once you find the result of the integral, treat this as an ordinary function of $R$ for which you can now take the limit $R \to \infty$.

1.4 Exercise. [10 points] Find the linear approximation of the function the function $\exp : \mathbb{R} \to \mathbb{R}$ near zero, and state how small does the argument have to be for a given level of precision.

1.5 Exercise. [30 points] Consider the function $f : \mathbb{R} \to \mathbb{R}$ 

$$ f(x) := \begin{cases} 
\sin(x) & x \geq 0 \\
x & x < 0 
\end{cases} (x \in \mathbb{R}). $$ 

1. [6 points] Determine where $f$ is continuous. 
2. [6 points] Determine where $f$ is differentiable, and where it is, calculate its derivative $f'$. 
3. [6 points] Determine where $f'$ is continuous. 
4. [6 points] Determine where $f'$ is differentiable, and where it is, calculate its derivative $f''$. 
5. [6 points] Determine where $f''$ is continuous.

2 Extra credit

2.1 Exercise. [20 points] Give an example of a differentiable function $f : [a, b] \to \mathbb{R}$ which is convex and $f'$ is monotone decreasing.

2.2 Exercise. [20 points] Using the product rule of differentiation and the fundamental theorem of calculus, prove the integration by parts theorem: If $f, g$ are two differentiable functions $[a, b] \to \mathbb{R}$ whose derivatives are integrable, then 

$$ \int_{a}^{b} f g' = f g \bigg|_{a}^{b} - \int_{a}^{b} f' g. $$

2.3 Exercise. [20 points] Let $n \in \mathbb{N}$ be given such that $n \geq 2$. Use integration by parts to find a formula connecting 

$$ \int_{a}^{b} (\cos(x))^n \, dx $$

and 

$$ \int_{a}^{b} (\cos(x))^{n-2} \, dx $$

Hint: Along the way, it will be useful to use $\sin^2 + \cos^2 = 1$. 
