Radiation Dominated Universe

We want to model a homogeneous and isotropic universe (like the Friedmann model) but where matter is modeled not as dust (as before) but rather as a radiation field, that is,

$$ T = \frac{8}{3} (4\pi u^0 u^0 - g) \quad (\text{HW6Q1}) $$

where $\epsilon$ is the energy density and $u^0$ is the velocity field w.r.t. which the isotropy applies: $u^0 = \frac{\partial}{\partial t}$, $\frac{\partial}{\partial t}$ in a co-moving chart. By homogeneity, $\epsilon$ depends only on time (in a co-moving chart).

Since $g$ is of the form $g = dt^2 - a(t)^2 dx_1 dx_2 dx_3$, a metric on the spacetime 3-mnfold at a given instant of time, solving the EFE $G_{ij} = 8\pi T$ \iff solving an equation for $a$ and $\epsilon$ (the Friedmann eqns):

LHS of EFE \rightarrow RHS of EFE

\[
\begin{align*}
\frac{a(\dot{a}^2 + k)}{3} - \frac{1}{3} \Lambda a^3 &= \frac{1}{2} \epsilon a^3 \\
2a\ddot{a} + \dot{a}^2 + k - \Lambda a^2 &= -\rho a^2
\end{align*}
\] (6.14) (6.15)

i) $\frac{1}{3} \epsilon a^4$ is conserved.

pf: \[
\left( \frac{4}{3} \epsilon a^3 \right) = a(\dot{a}^2 + k) + 2a\dot{a}\ddot{a} - \Lambda a^2 \dot{a}
\] (6.14) = \dot{a}(\dot{a}^2 + k + 2a\ddot{a} - \Lambda a^2) (6.15) = \dot{a}(-\rho a^2)

\text{HW6Q1} \Rightarrow (\frac{4}{3} \epsilon a^3) = \rho = -\frac{1}{3} \epsilon (a^3) \quad \rho = \frac{\rho}{3} \epsilon \frac{1}{3} (a^3)

\text{with } w = \frac{1}{3}

\Rightarrow (\frac{1}{3} \epsilon a^3) + \rho = \frac{1}{3} \epsilon (a^3) = 0

But $\left( \frac{1}{3} \epsilon a^3 \right) + \rho = \frac{1}{3} \epsilon (a^3) = \frac{1}{3} \dot{\rho} = \frac{1}{3} \epsilon (a^3) \left( 4\pi \Theta + \frac{1}{3} \epsilon (a^3) \right)$.
Indeed, \[ a \left( E a^3 \right) = \frac{d}{dt} \left( E a^3 \right) = \frac{d}{dt} \left( E a^3 \right) + 3 \frac{a}{a^3} \frac{d}{dt} \left( E a^3 \right) - \frac{d}{dt} \left( E a^3 \right) \]
\[ = \frac{\left( E a^3 \right) \left( 1 - 2 \alpha \right)}{\alpha^2} \frac{d}{dt} \left( E a^3 \right) - \frac{d}{dt} \left( E a^3 \right) \]
\[ \Rightarrow E a^{3(w+1)} = E q \quad \forall q \epsilon R \quad \text{if} \quad a \neq 0. \]
For us \( w = \frac{1}{2} \), so \( E a^4 = q \).

For photons, there is a grow. redshift \((HW6Q3)\) which says that, e.g., \( \frac{D}{t} = \frac{ac(t)}{ac(t)} \) \((6.10)\),

\( D \) is \( \mathcal{E} \) to the photon energy.

If the universe is expanding, there is a conserved # of photons distributed over a volume increasing as \( a^2 \) (that's the factor in the spatial metric) and then to get \( E \) we take another factor of \( a \) to take the redshift into account.

\( \text{(ii) Assume } a = 0. \text{ (radiation dominated universe).} \)

Then we already computed \( E a^4 = q \quad \forall q \epsilon R. \)

Note \((6.15)\) says \( \frac{1}{a} \left( \frac{4}{3} E a^3 \right) = -W E a^2 \), and by (i) this is equivalent to \( E a^4 = q \). So we only have to solve \((6.14): \)
\[ a \left( E a^3 \right) = \frac{1}{3} E a^3 \]
\[ \iff \dot{a}^2 + k - \frac{1}{3} E a^2 = 0 \]

\[ \iff \dot{a}^2 + k - \frac{a^2}{q^2} = 0 \]

Case \( k \neq 0 \)

Then \( \dot{a}^2 = \frac{a^2}{q^2} \iff a^2 \dot{a}^2 = q^2 \iff a \dot{a} = \pm q \)
\[ \iff \frac{1}{2} (a^2)' = \pm q \iff a^2(t) = \pm 2q t + q' \]
\[ \iff a(t) = \sqrt{\pm 2q t + q'} \]
Note that to get a big bang at time zero, \( t = 0 \), we can set \( \alpha(0) = 0 \). Then \( \dot{\alpha}(\infty) \), i.e. a big bang, \( \Rightarrow \Box^+ \). Since \( \alpha \in \mathbb{R} \), this implies we must take the + solution,
\[
\dot{\alpha}(0) = \sqrt{2\kappa t^2}
\]

Case 2: \( \beta_1 = 1 \)

Define \( \eta(t) = \int_0^t \frac{1}{a'} \, dt \) \( (\text{"conformal time"}) \)

Let \( f: [0, \infty) \to [0, \infty) \) be the inverse of \( \eta \). That is, for \( t \geq 0 \) or \( f(\eta(t)) = t \) \( \forall t \).
(We'll see it really exists \( \bigstar \))

Define \( \dot{\alpha}' = a \circ f' \).

Then \( \dot{\alpha}' = (a \circ f')(f'(\eta)) \dot{\eta} \) \( \text{chain rule} \).

Differentiate both sides at \( \eta = 0 \) to get
\[
(f' \circ \eta) \dot{\eta} = 1
\]
\[
\dot{\eta} = a^{-1} \quad \text{by Leibniz formula.}
\]
\[
\Rightarrow f' \circ \eta = a \quad \Rightarrow f' = a \circ f \quad \text{by } \eta \circ f = 1.
\]

We find \( \dot{\alpha}' = (a \circ f')(a \circ f) = (a \circ f)^2 \).

Thus the eqn \( \dot{\alpha}^2 + k - \frac{a^2}{\alpha^2} = 0 \) becomes
\[
(a \circ f)^2 + a^2 - q^2 = 0
\]

Composing it with \( f \) we get:
\[
(\dot{\alpha}')^2 + (\dot{\alpha})^2 \frac{a^2}{\alpha^2} - q^2 = 0
\]

Note also the initial value for \( \dot{\alpha} \):

We've chosen \( a(0) \equiv 0 \), and \( \eta(0) = 0 \Rightarrow f(0) = 0 \).

\[
\dot{\alpha}(0) = \alpha(\eta(0)) = \alpha(0) \equiv 0.
\]
\[
\dot{\alpha}' = \pm \sqrt{q^2 - k \alpha^2}
\]
\[
\frac{\dot{\alpha}'}{\sqrt{q^2 - k \alpha^2}} = \pm 1.
\]
\[
\int_0^1 \frac{1}{\sqrt{q^2 - kx^2}} = \pm \pi + q^1
\]

Note that if \( g(x) = \sqrt{q^2 - kx^2} \), then the anti-derivative of \( g \) is

\[
\int \frac{1}{\sqrt{q^2 - kx^2}} \, dx = \frac{1}{\sqrt{k}} \arctg \left( \frac{\sqrt{k}x}{\sqrt{q^2 - kx^2}} \right) = \frac{1}{\sqrt{k}} \arctg \left( \frac{k^x}{\sqrt{q^2 - kx^2}} \right) = G(x)
\]

Indeed, \( \arctg' \left( \frac{x}{\sqrt{1 + x^2}} \right) = \frac{1}{1 + x^2} \). Then

\[
G'(x) = \frac{1}{\sqrt{k}} \arctg' \left( \frac{\sqrt{k}x}{\sqrt{q^2 - kx^2}} \right) = \frac{1}{\sqrt{k}} \cdot \frac{1}{1 + \frac{kx^2}{q^2 - kx^2}} \cdot \sqrt{k^2 - kx^2}
\]

\[
= \frac{q^2 - kx^2 + kx^2}{(q^2 - kx^2) \sqrt{q^2 - kx^2}} = \frac{1}{\sqrt{q^2 - kx^2}} = g(x)
\]

\[
\Rightarrow \int_0^1 \frac{\bar{a}'}{\sqrt{q^2 - k\bar{a}^2}} = \int_0^1 g(\bar{a}) \, d\bar{a} = \int_0^1 G'(\bar{a}) \, d\bar{a} = \int_0^1 (G \circ \bar{a})' = G \circ \bar{a} \bigg|_0^1
\]

\[
(G \circ \bar{a}) (\bar{c}) = \pm \pi + q^1 \quad \text{as} \quad G(0) = 0.
\]

\[
\Rightarrow \bar{a}(\bar{c}) = G^{-1} (\pm \pi + q^1) \Rightarrow q^1 = 0
\]

\text{Case 2.1: } k=1 \Rightarrow G(x) = \arctg \left( \frac{x}{\sqrt{q^2 - x^2}} \right)

\[
\Rightarrow G^{-1}(x) = \pm q \sin (x)
\]

\[
\Rightarrow \bar{a} = q \sin \quad \text{(other choice of sign imphysical)}.
\]

\text{Case 2.2: } k=-1 \Rightarrow G(x) = -i \arctg \left( \frac{x}{\sqrt{q^2 + x^2}} \right)

\[
\Rightarrow G^{-1}(x) = \pm q \sinh (x)
\]

\[
\Rightarrow \bar{a} = q \sinh \quad \text{(the other choice of sign is imphysical)}.
\]

\* We now return to the question of the existence of \( a^{-1} \).

We saw \( \eta' = a^{-1} \), and we assume \( \eta(0) \) and a cont.
We have $\tilde{a}$ but we need $a$.
\[ f' = a' \tilde{a} \]
\[ f = \int \tilde{a} + q' = \frac{h}{q} \cos h + q' = q(1 - \cos h) \]

The constant $V$ is found by the constraint $f(0) = 0$.

Now that we have $f$ (independent of $a$) we can invert it to get
\[ a(t) = (\tilde{a} \circ \eta)(t) = \frac{h}{q} \sqrt{1 - (1 - \frac{1}{q^2})^2} \]

\[ \overset{\text{Case 1}}{\Rightarrow} \begin{cases} \eta(t) = \arccos \left(1 - \frac{1}{q^2}\right) + 2\pi n \quad \text{if } h > 0 \\ \eta(t) = \arccosh \left(1 - \frac{1}{q^2}\right) + 2\pi n \quad \text{if } h < 0 \end{cases} \quad \text{if } n \in \mathbb{N} \]

The Casual Structure of the Friedmann Models

Start with the Ansatz metric $g = dt^2 - a(t)^2 g_0$.

Switch to conformal time coordinates as above:
\[ \eta(t) = \int_{t_0}^{t} a^{-1} dt \quad \forall t \in [0, \infty) \]

\[ d\eta = \eta(t) dt = a^{-1} dt \]

\[ dt = a d\eta \]

\[ g = \tilde{a}^2 d\eta^2 - \tilde{a}^2 g_0 \quad \text{with coordinates } (\eta, x, \theta, \phi). \]

With $g_0$ as in (6.6) $(R_0 = 1, h = 1)$:
\[ g_0 = dt^2 + \sin^2(\alpha) d\phi^2 + \sin^2(\beta) d\phi^2 \quad \text{metric for } S^3 \cong \mathbb{R}^4 \quad \text{with} \]

with the coordinates $(x, \theta, \phi) \in [0, \pi] \times [0, 2\pi]^2$. 8 angles $(\chi, \theta, \phi)$. 

\[ (2) \]
For MD (matter dom.) and RD (radiation dom.) we want to compute the range of $\eta$ s.t. $\mathbf{a}$ goes from 0 (the big bang, its initial pt.) to 0 again (the big crunch, in case this happens for finite times).

MD: By (6.21), $\mathbf{a} \propto 1 - \cos(\eta)$ with $\eta \in [0, 2\pi]$ so that at $\eta = \pi$ we get zero again a big crunch.

RD: By the earlier exercise, $\mathbf{a} \propto \sin(\eta)$, so that at $\eta = \pi$ we get zero again.

ii) Is it possible to send a light signal from $(x, \eta) = (0, 0)$ to $(x, \eta) = (0, 0)$ in either case MD, RD before the end of the universe?

Note that geodesics starting at $x=0$ "see" the metric $g_0 = d\chi^2$.

Since the geodesics on the 3-sphere are great-circles (just as is the case for the 2-sphere), the geodesics correspond to fixed $\theta, \psi$ and varying $\chi \in [0, \pi]$. (This is not a proof).

Thus along geodesics starting at $(x, \eta) = (0, 0)$, we "see" the metric $g = 5(\eta)(d\eta^2 - d\chi^2)$ which is "conformally" equivalent to the Minkowski metric (prove this) so that null geodesics propagate along straight lines at angle $45^\circ$, just as in the Minkowski case.

$\Rightarrow$ In MD there's enough time (the whole duration of the universes lifetime) for a null geodesic to go in a loop, whereas in RD, the null geodesic can only cover half a distance.