Q1 Radiation Dominated Universe

We want to model a homogeneous and isotropic universe (like the Friedmann model) but where matter is modeled not as dust (as before) but rather as a radiation field, that is, $T = \frac{\varepsilon}{3}(4u^\flat \otimes u^\flat - g)$ (HW6Q1)

where ε is the energy density and u is the velocity field w.r.t. which the isotropy applies: $u^\mu = \delta^\mu{}_0$ in a co-moving chart. By homogeneity, ε depends only on time (in a co-moving chart).

Since g is of the form $g = dt^2 - a(t)^2 g_0$ $\exists g_0$ a metric on the spacelike 3-mnfld at a given instance of time, solving the E.F.E $G = 8\pi T \iff$ solving an equation for a and ε (the Friedmann eq-ns):

LHS of EFE RHS of EFE

$$\begin{cases} a(\dot{a}^2 + k) - \frac{1}{3}\Lambda a^3 = \frac{1}{3}\varepsilon a^3 & (6.14) \\ 2a\ddot{a} + \dot{a}^2 + k - \Lambda a^2 = -pa^2 & (6.15) \end{cases}$$

i) **Cl:** $\frac{1}{3}\varepsilon a^4$ is conserved.

Pf:

$$\left(\tfrac{1}{3}\varepsilon a^3\right)^\cdot \overset{(6.14)}{=} \dot{a}(\dot{a}^2 + k) + a2\dot{a}\ddot{a} - \Lambda a^2 \dot{a}$$

$$= \dot{a}(\dot{a}^2 + k + 2a\ddot{a} - \Lambda a^2)$$

$$\overset{(6.15)}{=} \dot{a}(-pa^2)$$

$$= -pa^2\dot{a} = -p\,\tfrac{1}{3}(a^3)^\cdot$$

HW6Q1 $p = w\varepsilon$ with $w = \frac{1}{3}$

$$= -w\varepsilon\,\tfrac{1}{3}(a^3)^\cdot$$

$$\Rightarrow \left(\tfrac{1}{3}\varepsilon a^3\right)^\cdot + w\varepsilon\,\tfrac{1}{3}(a^3)^\cdot = 0$$

But $\left(\frac{1}{3}\varepsilon a^3\right)^\cdot + w\varepsilon\,\frac{1}{3}(a^3)^\cdot = \frac{1}{3}a^{-3w}\left(\varepsilon a^{3(1+w)}\right)^\cdot$

Indeed, $a \; (\varepsilon a \;\;) \; =$

$$= \cancel{a^{-3w}}(\varepsilon a^3)^\cdot \cancel{a^{3w}} + a^{-3w}\varepsilon a^3\, 3w\, a^{3w-1}\dot a$$

$$= (\varepsilon a^3)^\cdot + \underbrace{3w\varepsilon a^2 \dot a}_{w\varepsilon (a^3)^\cdot}$$

$$\Rightarrow \quad \varepsilon a^{3(w+1)} = q \quad \exists\, q \in \mathbb{R}, \text{ if } a \neq 0.$$

For us $w = \frac{1}{3}$, so $\varepsilon a^4 = q$

For photons, there is a grav. redshift (HW6Q3) which says that, e.g. $\frac{\nu_2}{\nu_1} = \frac{a(t_1)}{a(t_2)}$ (6.10)

ν is $\propto$ to the photon energy.

If the universe is expanding, there is a conserved # of photons distributed over a volume increasing as a^3 (that's the factor in the spatial metric) and then to get ε we take another factor of a to take the redshift into account.

ii) Assume $\Lambda = 0$ (radiation dominated universe).

Then we already computed $\varepsilon a^4 = q \quad \exists\, q \in \mathbb{R}$.

Note (6.15) says $\frac{1}{\dot a}\left(\frac{1}{3}\varepsilon a^3\right)^\cdot = -w\varepsilon a^2$, and by (i) this is equivalent to $\varepsilon a^4 = q$. So we only have to solve (6.14): $a(\dot a^2 + k) = \frac{1}{3}\varepsilon a^3$

$$\Leftrightarrow \quad \dot a^2 + k - \frac{1}{3}\varepsilon a^2 = 0$$

$$\Leftrightarrow \quad \boxed{\dot a^2 + k - \frac{q^2}{a^2} = 0}$$

Case 1: $k = 0$

Then $\dot a^2 = \frac{q^2}{a^2} \Leftrightarrow a^2 \dot a^2 = q^2 \Leftrightarrow a\dot a = \pm q$

$$\Leftrightarrow \frac{1}{2}(a^2)^\cdot = \pm q \Leftrightarrow a^2(t) = \pm 2qt + q'$$

$$\Leftrightarrow a(t) = \sqrt{\pm 2qt + q'}$$

Note that to get a big bang at time zero, we can set $a(0)=0$. Then $\dot{a}(\infty)$, i.e. a big bang. $\Rightarrow \boxed{q'=0}$. Since $a\in\mathbb{R}$, this implies we must take the + solution,

$$\boxed{a(t)=\sqrt{2qt}}$$

Case 2: $|k|=1$

Define $\eta(t) := \int_{[0,t]} a^{-1} \quad \forall\, t\in[0,\infty)$ ("conformal time")

Let $f:[0,\infty)\to[0,\infty)$ be the inverse of η. That is,
$f\circ\eta := \mathbb{1}$ or $f(\eta(t))\equiv t \quad \forall\, t$.
(We'll see it really exists ⊛)

Define $\tilde{a} := a\circ f$.

Then $\tilde{a}' = (a'\circ f)f' \equiv (\dot{a}\circ f)f'$

chain rule

Differentiate both sides of $f\circ\eta=\mathbb{1}$ to get

$(f'\circ\eta)\eta' = 1$

$\eta' = a^{-1}$ by Leibniz formula.

$\Rightarrow f'\circ\eta = a \quad \Rightarrow f' = a\circ f$ by $\eta\circ f\equiv\mathbb{1}$.

We find $\tilde{a}' = (\dot{a}\circ f)(a\circ f) \equiv (\dot{a}a)\circ f$

Thus the eq-n $\dot{a}^2 + k - \frac{q^2}{a^2} = 0$ becomes

$$(a\dot{a})^2 + a^2k - q^2 = 0$$

Composing it with f we get:

$$(\tilde{a}')^2 + (\tilde{a})^2 k - q^2 = 0$$

Note also the initial value for $\tilde{a}$:

We've chose $a(0)\equiv 0$, and $\eta(0)=0 \Rightarrow f(0)=0$.

$$\tilde{a}(0) \equiv a(f(0)) = a(0) \equiv 0.$$

$$\tilde{a}' = \pm\sqrt{q^2 - k\tilde{a}^2}$$

$$\frac{\pm\tilde{a}'}{\sqrt{q^2-k\tilde{a}^2}} = \pm 1$$

$$\int_0^\tau \frac{\tilde{a}'}{\sqrt{q^2 - k\tilde{a}^2}} = \pm\tau + q'$$

Note that if $g(x) = \frac{1}{\sqrt{q^2 - kx^2}}$ then the anti-derivative of g is $\int \frac{1}{\sqrt{q^2-kx^2}}dx = \frac{1}{\sqrt{k}} \operatorname{arctg}\left(\frac{\sqrt{k}x}{\sqrt{q^2-kx^2}}\right) =: G(x)$

Indeed, $\operatorname{arctg}'(\alpha) = \frac{1}{1+\alpha^2}$. Then

$$G'(x) = \frac{1}{\sqrt{k}} \operatorname{arctg}'\left(\frac{\sqrt{k}x}{\sqrt{q^2-kx^2}}\right) = \frac{1}{\cancel{\sqrt{k}}} \frac{1}{1+\frac{kx^2}{q^2-kx^2}} \times$$

$$\times \frac{\cancel{\sqrt{k}}\sqrt{q^2-kx^2} - \cancel{\sqrt{k}}x\cancel{\tfrac{1}{2}}(q^2-kx^2)^{-1/2}(-k)\cancel{2}x}{q^2-kx^2}$$

$$= \frac{q^2 \cancel{-kx^2} + \cancel{kx^2}}{(q^2 \cancel{-kx^2} + \cancel{kx^2})\sqrt{q^2-kx^2}} = \frac{1}{\sqrt{q^2-kx^2}} = g(x)$$

$$\Rightarrow \int_0^\tau \frac{\tilde{a}'}{\sqrt{q^2-k\tilde{a}^2}} = \int_0^\tau g(\tilde{a})\tilde{a}' = \int_0^\tau G'(\tilde{a})\tilde{a}'$$

$$= \int_0^\tau (G\circ\tilde{a})' = G\circ\tilde{a}\Big|_0^\tau$$

$$\Rightarrow (G\circ\tilde{a})(\tau) = \pm\tau + q' \qquad \text{as } G(0)=0.$$

$$\Downarrow$$

$$G^{-1}(0)=0$$

$$\Rightarrow \tilde{a}(\tau) = G^{-1}(\pm\tau + q') \Rightarrow q' = 0$$

Case 2.1: $k=1 \Rightarrow G(x) = \operatorname{arctg}\left(\frac{x}{\sqrt{q^2-x^2}}\right)$

$$\Rightarrow G^{-1}(x) = \pm q\sin(x)$$

$$\Rightarrow \boxed{\tilde{a} = q\sin}$$ (other choice of sign unphysical).

Case 2.2: $k=-1 \Rightarrow G(x) = -i\operatorname{arctg}\left(\frac{ix}{\sqrt{q^2+x^2}}\right)$

$$\Rightarrow G^{-1}(x) = \pm q\sinh(x)$$

$$\Rightarrow \boxed{\tilde{a} = q\sinh}$$ (the other choice of sign is unphysical).

(*) We now return to the question of the existence of f:

We saw $\eta' = a^{-1}$, and we assume $a>0$ and a cont.

and less than ∞ $\Rightarrow$ η is cont. diff. and non-zero

$\Rightarrow$ η is invertible by the inverse function theorem.

We have $\tilde{a}$ but we need a.

$$f' = a \circ f \equiv \tilde{a}$$

$$\Rightarrow f = \int \tilde{a} + q' = \begin{cases} \xrightarrow{k=1} -q\cos + q' = q(1-\cos) \\ \xrightarrow{k=-1} q\cosh + q' = q(\cosh - 1) \end{cases}$$

The constant q' is found by the constraint $f(0)=0$.

Now that we have f (indep. of a) we can invert it to get η (indep. of a):

$$\eta(t) = \begin{cases} \xrightarrow{k=1} \pm\arccos\left(1-\frac{t}{q}\right) + 2\pi n \\ \pm\operatorname{arccosh}\left(1-\frac{t}{q}\right) + 2\pi i n \end{cases} \qquad \exists\, n \in \mathbb{Z}$$

$$\Rightarrow \boxed{a(t) = (\tilde{a}\circ\eta)(t) = \begin{cases} \xrightarrow{k=1} \sqrt{1-\left(1-\frac{t}{q^2}\right)^2} \\ \xrightarrow{k=-1} \sqrt{\left(1-\frac{t}{q}\right)^2 - 1} \end{cases}}$$

Q2 The Casual Structure of the Friedmann Models

Start with the Ansatz metric $g = dt^2 - a(t)^2 g_0$.

Switch to conformal time coordinates as above:

$$\eta(t) := \int_{[0,t]} a^{-1} \qquad \forall\, t \in [0,\infty)$$

$$d\eta = \eta'(t)\,dt = a^{-1}\,dt$$

$$\Rightarrow dt = a\,d\eta$$

$$\Rightarrow g = \tilde{a}^2\,d\eta^2 - \tilde{a}^2 g_0$$ with coordinates $(\eta, \chi, \theta, \varphi)$.

With g_0 as in (6.6) $(R_0 \equiv 1, k=1)$:

$$g_0 = d\chi^2 + \sin(\chi)^2\left(d\theta^2 + \sin(\theta)^2\,d\varphi^2\right)$$ metric for $S^3 \subseteq \mathbb{R}^4$ with 3 angles (χ, θ, φ).

with the coordinates $(\chi, \theta, \varphi) \in [0,\pi] \times [0, 2\pi]^2$.

6

i) For MD (matter dom.) and RD (radiation dom.) we want to compute the range of η s.t. $\tilde{a}$ goes from 0 (the big bang, its initial pt.) to 0 again (the big crunch, in case this happens for finite times).

<u>MD</u>: By (6.21), $\tilde{a} \propto 1-\cos(\eta)$ with $\eta \in (0, 2\pi)$ so that at $\eta = 2\pi$ we get zero again, a big crunch.

<u>RD</u>: By the earlier exercise, $\tilde{a} \propto \sin(\eta)$, so that at $\eta = \pi$ we get zero again.

ii) Is it possible to send a light signal from $(\chi, \eta) = (0,0)$ to $(\chi, \eta) = (0,0)$ in either case MD, RD before the end of the universe?

Note that geodesics starting at $\chi = 0$ "see" the metric

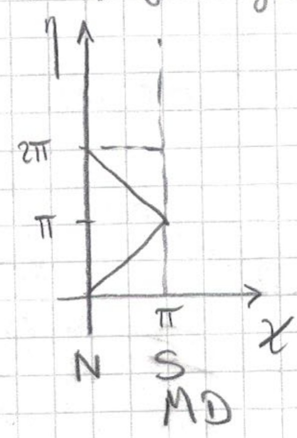
$$g_0 = d\chi^2$$

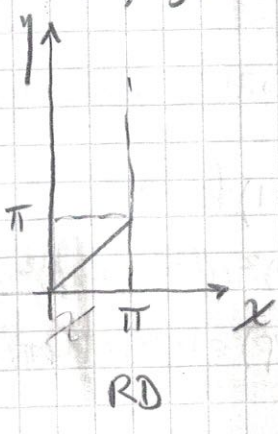
Since the geodesics on the 3-sphere are great-circles (just as is the case for the 2-sphere), the geodesics correspond to fixed θ, φ and varying $\chi \in [0, \pi]$. (this is not a proof).

Thus along geodesics starting at $(\chi, \eta) = (0,0)$, we "see" the metric

$$g = \tilde{a}(\eta)(d\eta^2 - d\chi^2)$$

which is "conformally" equivalent to the Mikowski metric (prove this) so that null geodesics propagate along straight lines of angle $\pm 45°$, just as in the Minkowski case.





$\Rightarrow$ In MD there's enough time (the whole duration of the universe's lifetime) for a null geodesic to go in a loop, whereas in RD, the null geodesic can only cover half a distance.