GR — HW11 — 3/12/2017

Q1 The Magnitude-Redshift Relation

Energy flux: energy per unit time per unit surface, describing the flow of energy through a region of space.

Light intensity: energy per unit time, describing the intensity of a source.

Clearly in flat spacetime, a light source of intensity L has flux f at distance d given by

$$f = \frac{L}{4\pi d^2}$$

i) Q: In the Friedmann spacetime the corresponding relation is

$$f = \frac{L}{4\pi d^2 (1+z)^2} \left(\frac{\chi}{\sin(\chi)} \right)^2 \quad \text{as in (6.5)}$$

$$= \frac{L}{4\pi d^2 (1+z)^2} \left(1 - \frac{\Omega_k}{3} (Hd)^2 + \dots \right)$$

where d is the proper distance ($\equiv$ the present distance between the source at spatial origin and the observer at some radial location on the 3-sphere specified by χ. z is the redshift parameter (as in pp. 53) given by $z :\equiv \frac{\nu_1}{\nu_2} - 1$ with ν_i the freq. at the source ($i=1$) or the destination ($i=2$). H is the Hubble constant given by $H \equiv \frac{\dot{a}(t_0)}{a(t_0)}$ (t_0 the age of the universe, a its expansion rate) and $\Omega_k \equiv 1 - \Omega_\Lambda - \Omega_m$

$\Omega_\Lambda \equiv \frac{\Lambda}{3H^2}$, $\quad \Omega_m \equiv \frac{\rho(t_0)}{3H^2}$ ← matter density at time t_0.

Pf.: Think of the energy flux in terms of a stream of photons. Due to grav. redshift their freq. ν_i differ between source ($i=1$) and destination ($i=2$). The proper times between successive photons $\Delta\tau_i$ at source or destination also differ due to grav. redshift.

$k \in \{0, \pm 1\}$!

Since we are using the metric (6.6), ~~thinking of its space time metric as that of a 3 sphere with radius 1~~, the distance between the origin and the point χ is simply χ, and the area over which energy is distributed is $4\pi \sin(\chi)^2$. (here $\sin(\chi)$ is like the radial coordinate on S^2).

By defn, $\nu_2 = \nu_1 (1+z)^{-1}$. We also know

$\Delta\tau_2 = \Delta\tau_1$ bcs. of (6.9) (see sol-n of HW6Q3)

In a (cosmic) time interval Δt the # of photons emitted at the source is

$$\Delta t \times \left(\text{rate of photon emitting}\right) =$$

$\Rightarrow$ Their energy is

$$\left(\text{energy of one photon}\right) \times \left(\#\text{ of photons}\right) = \quad \equiv L\Delta t$$

Similarly the energy at the destination is

We find $f \equiv \frac{\text{energy at destination}}{\Delta t \quad \text{area}} =$

$=$ [redacted]

This gives us the 1^{st} line to be proven.
For the 2^{nd} line, use

$$\mathrm{sinn}(x) \approx x - \frac{1}{6} k x^3 + O(x^5)$$

$$\rightarrow \left(\frac{x}{\mathrm{sinn}(x)}\right)^2 \approx$$ [redacted]

But according to (6.26), $k = -\mathrm{sgn}\,\Omega_k$ and $|\Omega_k|^{1/2} = H^{-1} \Rightarrow k = -\frac{\Omega_k}{|\Omega_k|} = -\Omega_k H^2$

and $x = d$.

$$\Rightarrow kx^2 =$$ [redacted]

ii) Cl.: With q being the dimensionless deceleration parameter

$$q \equiv -a(t_0)\,\ddot{a}(t_0)\,\dot{a}(t_0)^{-2}$$

as in pp. 61 we find

$$f = \frac{LH^2}{4\pi z^2}\left(1 - (1-q)z + O(z^2)\right)$$

Pf.: The last eq-n on pp. 61 gives the distance-redshift relation:

$$z = Hd + \frac{1}{2}(1+q)(Hd)^2 + \ldots$$

Solving this for d we find by iteration

$d =$ [redacted]

Hence $d^2(1+z)^2 =$ [redacted]

$\Rightarrow \rho =$ [redacted]

Q2 Killing Vectors

Recall that a Killing vector field is a vector field K s.t. $\mathcal{L}_K(g)=0$, with $\mathcal{L}$ being the Lie derivative and g the metric.

Cl.: A Killing vector obeys a one-param. gp. of isometries ϕ_t and vice versa any vector field whose flow generates a 1-param. gp. of isom. is a Killing vector.

Pf.: Recall ϕ_t is an isometry $\Leftrightarrow$ $g = \phi_t^* g$ with ϕ_t^* being the pull-back.

$$\mathcal{L}_K(g) \equiv \partial_t\big|_{t=0} \phi_t^* g$$

Note if ∇ is the covariant derivative assoc. w/ g then

$$(\mathcal{L}_X(g))(Y_1, Y_2) = (\nabla X^{\flat})(Y_1, Y_2) + (\nabla X^{\flat})(Y_2, Y_1)$$

(Wald eq-n (C.2.16))

Hence Killing's eq-n is $\nabla_a K_b + \nabla_b K_a = 0$ (abst. index notation of Wald)

Note that (as explained in Wald) $\exists$ max. of 10 Killing vector fields on a 4-dim spacetime.

i) **Cl.:** In Minkowski spacetime all 10 Killing vector fields correspond to all conserved quantities as 4 transl, generators, ~~...~~ 3 rotation generators and 3 boost generators.

Pf.: By the Killing eq-n in flat spacetime we find for the components of any Killing vector K in a chart, K^μ, that $\partial_\nu K_\mu + \partial_\mu K_\nu = 0$ (16 eq-ns $\forall\ (\mu,\nu)$)

$\Rightarrow \partial_\sigma \partial_\nu K_\mu =$

Do this twice to find $\partial_\sigma \partial_\nu K_\mu = -$,

That is, $\boxed{\quad = 0}$.

$\Rightarrow K_\mu(x) =$ $\quad \forall\ x \in \mathbb{R}^4$

ii) **Cl.:** In a spacetime w/ coord. (u, v, x, y) and metric

$$ds^2 = (du\,dv + dv\,du) - a(u)^2 dx^2 - b(u)^2 dy^2$$

$\exists\ a, b : \mathbb{R} \to \mathbb{R}$, there are <u>five</u> lin. indep. Killing vectors:

$$K_{(1)}{}^\mu = (0,1,0,0)$$
$$K_{(2)}{}^\mu = (0,0,1,0)$$
$$K_{(3)}{}^\mu = (0,0,0,1)$$
$$K_{(4)}{}^\mu(u,v,x,y) = (0, x, \textstyle\int a^{-2}, 0)$$
$$K_{(5)}{}^\mu(u,v,x,y) = (0, y, 0, \textstyle\int b^{-2}\,du)$$

Pf.: