The Magnitude-Redshift Relation

Energy flux: energy per unit time per unit surface, describing the flow of energy through a region of space.

Light intensity: energy per unit time, describing the intensity of a source.

Clearly in flat spacetime, a light source of intensity $I$ has flux $f$ at distance $d$ given by

$$f = \frac{I}{4\pi d^2}$$

2) (a.5) In the Friedmann spacetime the corresponding relation is

$$f = \frac{L}{4\pi d^2 (Hd)^2} \left( \frac{\mu}{\sin(\mu)} \right)^2$$

$$= \frac{L}{4\pi d^2 (Hd)^2} \left( 1 - \frac{2k}{3} (Hd)^2 + O(1) \right)$$

where $d$ is the proper distance (= the present distance between the source at spatial origin and the observer at some radial location on the 3-sphere specified by $\chi$, $z$ is the redshift parameter (as in pp. 53) given by $z = \frac{d_i}{d} - 1$ with $d_i$ the freq. at the source ($i=1$) or the destination ($i=2$). $H$ is the Hubble constant given by $H = \frac{kd}{\mu^2}$ (to the age of the universe, $\mu$ its expansion rate) and $2k = 1 - \Omega_m - \Omega_k$.
\[ \Omega_A = \frac{\Delta}{3H^2} \quad \Omega_{m} = \frac{\rho (a_0)}{3H^2} \quad \text{matter density at time } t_0. \]

Think of the energy flux in terms of a stream of photons. Due to gravity, redshifts their frequencies differ between source and destination. The proper times between successive photons at source or destination also differ due to gravity, redshift.

Since we are using the metric \((6.6)\), thinking of its space-like metric as that of a 3-sphere with radius \(\frac{1}{a}\), the distance between the origin and the point \(x\) is simply \(\frac{1}{a}\), and the area over which energy is distributed is \(4\pi \sin(x) \left(\frac{\alpha}{2}\right)^2\) (where \(\sin(x)\) is like the radial coordinate on \(S^2\)).

By definition, \(\theta_2 = \theta_1 \left(1 + \frac{\Delta t}{a}\right)^{-1}\). We also know \(\Delta t_2 = \Delta t_1\), bcs. of \((6.9)\) (see solution of HW6Q2).

In a \(\Delta t\) time interval \(\Delta t\), the # of photons emitted at the source is

\[ \Delta t \times \text{(rate of photon-emitting)} = \]

\[ \Rightarrow \text{Their energy is} \]

\[ \left(\text{energy of one photon}\right) \times \left(\text{# of photons}\right) = \]

\[ \Rightarrow \text{Similarly, the energy at the destination is} \]

\[ \text{We find } f = \frac{\text{energy at destination}}{\Delta t \times \text{area}} = \]
This gives us the 1st line to be proven.

For the 2nd line, use
\[ \sin(x) \approx x - \frac{1}{6} x^5 + O(x^7) \]

\[ \Rightarrow \left( \frac{x}{\sin(x)} \right)^2 \approx 1 \]

But according to (6.26), \[ k = -\text{sign}(k) \text{ and } \]
\[ H_0 \sqrt{\Omega} = H^{-1} \Rightarrow \frac{k}{H_0} = \frac{\Omega H}{H_0^2} = -\Omega H^2 \]

and \( x = d \).

\[ \Rightarrow k d^2 = \frac{1}{\Omega H^2} \]

ii) \( \chi^{(i)} \) With \( \gamma \) being the dimensionless deceleration parameter
\[ \gamma = -\alpha(t_0) \frac{\dot{a}(t_0)}{a(t_0)} \frac{\dot{a}(t_0)}{a(t_0)} \]

as in pp. 61 we find
\[ f = \frac{H_0^2}{4\pi^2} \left( 1 - \frac{1}{2} \frac{\Omega H}{H_0^2} \right) \]

pp. 60: The last eqn on pp. 61 gives the distance-redshift relation:
\[ z = \frac{H_0}{1 + \frac{1}{2} (1+z)(H_0)^2} + 0(z^2) \]

Solving this for \( d \), we find by iteration
\[ d = \ldots \]
Hence \( d^2(1+z)^2 = \)

\[ \Rightarrow p = \]

### Killing Vectors

Recall that a Killing vector field is a vector field \( \xi \) s.t. \( \mathcal{L}_{\xi} g = 0 \), with \( \mathcal{L} \) being the Lie derivative and \( g \) the metric.

#### Cl. 1

A Killing vector field obeys a one-param. gp. of isometries \( \xi^t \) and vice versa any vector field whose flow generates a 1-param. gp. of isom. is a Killing vector.

#### Pf.

Recall \( \xi_t \) is an isometry \( \iff \xi = \xi^*_t g \) with \( \xi^*_t \) being the pull-back.

\[ \mathcal{L}_{\xi}(g) = \partial_t |_{t=0} \xi^*_t g \]

Note if \( \nabla \) is the covariant derivative assoc w/ \( g \) then

\[ (\mathcal{L}_{\xi}(g))(Y_1, Y_2) = (\nabla_{\xi})Y_1, Y_2 \]

([Wald eqn (C.216)])

Hence Killing's eqn is \( \nabla_{\xi} Y_1 + \nabla_{\xi} Y_2 = 0 \) (abst. index notation of Wald)

Note that (as explained in Wald) \( \exists \) max. of 10 Killing vector fields on a 4-dim spacetime.

Cl. 2

In Minkowski spacetime all 10 Killing vector fields correspond to all conserved quantities as 4 transl.
ii) O(4) In a spacetime w/ coord. \((t,x,y)\) and metric
\[ds^2 = (\text{di}t)^2 + \text{d}x^2 + \text{d}y^2 - \text{a}(t)^2 \text{d}x^2 - \text{b}(t)^2 \text{d}y^2\]
\(\forall \ a, b \in \mathbb{R} \to \mathbb{R}\), there are five lin. indep. Killing vectors:
\(K_0\)\(\mu\) = \((0,1,0,0)\)
\(K_x\)\(\mu\) = \((0,0,1,0)\)
\(K_y\)\(\mu\) = \((0,0,0,1)\)
\(K_{(x)}\)\(\mu\) = \((1,0,0,0)\)
\(K_{(y)}\)\(\mu\) = \((0,1,0,0)\)