General relativity. Problem set 1.

HS 17 *Due:* Tue, September 26, 2017

1. The sphere as a manifold

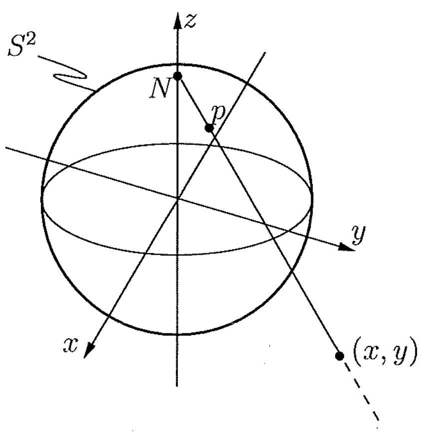
Consider the sphere

$$S^2 = \{p = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

and its covering $S^2 = U_+ \cup U_-$ by the two open sets

$$U_\pm = S^2 \setminus \{(0, 0, \pm 1)\}$$

obtained by removing the north, resp. south pole from S^2. The stereographic projection $p \mapsto (x, y)$ shown in the figure provides a chart for U_+ with coordinate neighborhood $\mathbb{R}^2 \ni (x, y)$; similarly there is one for U_- with neighborhood $\mathbb{R}^2 \ni (\bar{x}, \bar{y})$. On which subset of $\mathbb{R}^2$ is the transition function $(x, y) \mapsto (\bar{x}, \bar{y})$ defined? Compute that function.



2. Tensors

a) Show that not all tensors in

$$\begin{aligned} V \otimes V &= \{T \mid T \text{ is a bilinear form over } V^* \times V^*\} \\ &= \{\text{linear combinations of tensors } v_1 \otimes v_2 \mid v_1, v_2 \in V\} \end{aligned}$$

are of the form $v_1 \otimes v_2$ (simple tensors).

b) Identify $V \otimes W^*$ with the linear space $\mathcal{L}(W, V)$ of linear maps $W \to V$.

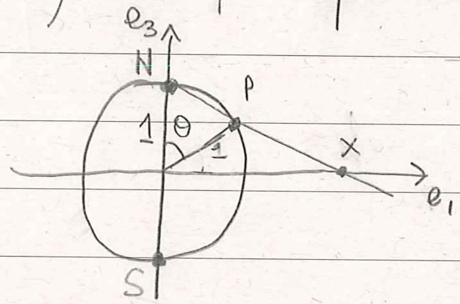
GR — HW #1 Solutions

Q1 The Sphere as a Manifold

$$S^2 := \{ x \in \mathbb{R}^3 \mid \|x\| = 1 \} = \underbrace{(\|\cdot\| - 1)^{-1}(\{0\})}_{\text{zero set}}$$

Define $\mathcal{U}_\pm := S^2 \setminus \{\mp e_3\}$ where $\{e_i\}_{i=1}^3$ is the std. basis of $\mathbb{R}^3$.

Define maps $\eta_\pm : \mathcal{U}_\pm \to \mathbb{R}^2$ by stereographic proj:



$\eta_- : p \mapsto x$ defined $\forall p \in S^2 \setminus \{N\}$.
given by the formula
$\eta_-(\theta) = \mathrm{ctg}(\theta/2)$ on the plane $y=0$.

Outside of the plane we rotate by the azimuthal angle φ:
$\eta_-(\theta,\varphi) = \mathrm{ctg}(\theta/2)\, e_r + \varphi\, e_\varphi$
We can then convert this to Cartesian coordinates:

$$\eta_-(x,y,z) = \mathrm{ctg}\left(\frac{\mathrm{arctg}\left(\frac{z}{\sqrt{x^2+y^2}}\right)}{2}\right)\left(\cos\left(\mathrm{arctg}\left(\tfrac{y}{x}\right)\right)e_1 + \sin\left(\mathrm{arctg}\left(\tfrac{y}{x}\right)\right)e_2\right) +$$

$$+ \mathrm{arctg}\left(\tfrac{y}{x}\right)\left(-\sin\left(\mathrm{arctg}\left(\tfrac{y}{x}\right)\right)e_1 + \cos\left(\mathrm{arctg}\left(\tfrac{y}{x}\right)\right)e_2\right)$$

Note the relations $\sin(\mathrm{arctg}(\alpha)) = \frac{\alpha}{\sqrt{1+\alpha^2}}$, $\cos(\mathrm{arctg}(\alpha)) = \frac{1}{\sqrt{1+\alpha^2}}$

$$\mathrm{ctg}\left(\tfrac{1}{2}\mathrm{arctg}(\alpha)\right) = \frac{1+\sqrt{1+\alpha^2}}{\alpha}, \quad x^2+y^2+z^2 = 1$$

to get $\eta_-(x,y,z) = \frac{1}{1-z}(x e_1 + y e_2) \quad \forall (x,y,z) \in S^2 \setminus \{N\}$.
Similarly, $\eta_+(x,y,z) = \frac{1}{1+z}(x e_1 + y e_2)$

Clm: $\mathcal{U}_\pm \in \mathrm{Open}(S^2)$ (Complements of closed maps)
Clm: $\eta_\pm$ are continuous.

$\boxed{2}$ The transition map is defined on the overlap $\mathcal{U}_+ \cap \mathcal{U}_-$. It is given by $\eta_+ \circ \eta_-^{-1} : \eta_-(\mathcal{U}_+ \cap \mathcal{U}_-) \to \eta_+(\mathcal{U}_+ \cap \mathcal{U}_-)$

Note $\eta_\pm(\mathcal{U}_+ \cap \mathcal{U}_-) \cong \mathbb{R}^2 \setminus \{0\}$.

$\eta_-^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathcal{U}_+ \cap \mathcal{U}_-$ is given by

$$x \longmapsto (1+\|x\|^2)^{-1}(2x_1 e_1 + 2x_2 e_2 + (-1+\|x\|^2)e_3)$$

Indeed, $\dfrac{4x_1^2 + 4x_2^2 + 1 - 2\|x\|^2 + \|x\|^4}{1+2\|x\|^2+\|x\|^4} = 1 \checkmark \Rightarrow \eta_-^{-1}(x) \in S^2 \checkmark$

$$(\eta_- \circ \eta_-^{-1})(x) = \frac{1}{\underbrace{1 - \frac{-1+\|x\|^2}{+1+\|x\|^2}}_{\frac{1+\|x\|^2}{1+\cancel{\|x\|^2}+1\cancel{-\|x\|^2}}}}\left(\frac{2x_1}{1+\|x\|^2}e_1 + \frac{2x_2}{1+\|x\|^2}e_2\right) = x \checkmark$$

Similarly $(\eta_-^{-1} \circ \eta_-)(x) = x \quad \forall\, x \in \mathcal{U}_+ \cap \mathcal{U}_-$

Then $$(\eta_+ \circ \eta_-^{-1})(x) = \frac{1}{\underbrace{1 + \frac{-1+\|x\|^2}{+1+\|x\|^2}}_{\frac{1+\|x\|^2}{\cancel{1}+\|x\|^2\cancel{-1}+\|x\|^2}}}\left(\frac{2x_1}{1+\|x\|^2}e_1 + \frac{2x_2}{1+\|x\|^2}e_2\right)$$

$$= \|x\|^{-2}x$$

Cl.: $\eta_+ \circ \eta_-^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is smooth.

$f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x) := \|x\|^2 - 1$ $\boxed{3}$

Def.: $x \in \mathbb{R}^3$ is a regular point of f iff $(df)_x : T_x\mathbb{R}^3 \to T_{f(x)}\mathbb{R}$ is surjective.

Def.: $x \in \mathbb{R}$ is a regular value of f iff $\forall\ y \in f^{-1}(\{x\})$, y is a regular point of f.

Cl.: 0 is a regular value of f.

Pf.: $f^{-1}(\{0\}) = S^2$

$(df)_i(x) = (\partial_i f)(x) = 2x_i$

Of course $2x_i : \underbrace{T_p\mathbb{R}^3}_{\cong \mathbb{R}^3} \to \underbrace{T_p\mathbb{R}}_{\cong \mathbb{R}}$ is surjective.

Cl.: If $f: M \to N$ is smooth and $x \in N$ is a regular value of f, then $f^{-1}(\{x\})$ is a submanifold of M of dimension $m-n$.

$\Rightarrow S^2$ is a submanifold of $\mathbb{R}^3$, with dimension 2.

4

Q2 Tensors

a) Cl.: $\{v_1 \otimes v_2 \in V \otimes V\} \not\supseteq V \otimes V$

Pf.: Let $T \in V \otimes V$ be given of rank 2 or higher.
Let $\{e_i\}_i$ be an ONB for V.
Then $T = T_{ij}\, e_i \otimes e_j$ $\exists$ $\{T_{ij}\}_{i,j} \subseteq \text{field}(V)$.
Assume $\exists$ $(v_1, v_2) \in V^2$; $T = v_1 \otimes v_2$.
Then $T = v_1 \otimes v_2 = (v_1)_i e_i \otimes (v_2)_j e_j = (v_1)_i (v_2)_j\, e_i \otimes e_j$
$\Rightarrow$ $T_{ij} = (v_1)_i (v_2)_j$
$\Rightarrow$ $T_{ij} e_j = (v_1)_i (v_2)_j e_j = (v_1)_i v_2$
$\Rightarrow$ Every i^{th} row of the matrix T, $T_{ij}e_j$, is a multiple of v_2, so T has rank 1. $\Rightarrow$ $\boxed{\perp}$

b) Cl.: $V \otimes W^* \cong \underbrace{\mathcal{L}(W, V)}_{\text{lin. maps } W \to V}$ as v/sp. isomorphism.

Pf.: Define $\eta : V \otimes W^* \to \mathcal{L}(W, V)$ by:

Let $T \in V \otimes W^*$ be given. Then $T = \sum_j v_j \otimes \eta_j^*$
for $\{v_j\}_j \subseteq V$, $\{\eta_j\}_j \subseteq W^*$.

Let $w \in W$ be given. Then $\sum_j v_j \otimes \underbrace{\eta_j^*(w)}_{\in \text{Field}(W)} \in V$

$$\Rightarrow \boxed{\eta(T) := \sum_j \eta_j^*(\cdot)\, v_j}$$