# GR – HW #2 Solutions – 27/9/2017

## Q1 The Jacobi Identity (Wald Ch. 2 Ex. 3)

Let $A, B, C$ be vector fields on the manifold $\mathcal{M}$.

*Repetition of script:*

That means they are sections in the tangent bundle $T\mathcal{M}$.
Let $f \in \mathcal{F}(\mathcal{M})$. For any $p \in \mathcal{M}$, $A(p) \in T_p\mathcal{M}$, so $A(p): \mathcal{F}_p(\mathcal{M}) \to \mathbb{R}$. Hence $(A(p))(f) \in \mathbb{R}$. Thus $p \mapsto (A(p))(f)$ is again in $\mathcal{F}(\mathcal{M})$.
Thus it makes sense to compose vector fields together, as $A \circ B \equiv A(p \mapsto (B(p))(\cdot))$ [A acts on the map whose argument is p]
However note that $A \circ B$ is not a tangent field since it doesn't have the Leibnitz property:

$$\begin{aligned}(AB)(fg) &\equiv A(p \mapsto (B(p))(fg)) \\ &= A(p \mapsto (B(p))(f)g(p) + f(p)(B(p))(g)) \quad \text{B Leibnitz} \\ &= A((p \mapsto (B(p))(f)g(p)) + (p \mapsto f(p)(B(p))(g))) \quad \text{A linear} \\ &= A(p \mapsto (B(p))(f)g(p)) + A(p \mapsto f(p)(B(p))(g)) \\ &\equiv A((B(\cdot))(f)g) + A(f(B(\cdot))(g)) \\ &= ((AB)(\cdot))(f)g(\cdot) + (B(\cdot))(f)(A(\cdot))(g) + (A(\cdot))(f)(B(\cdot))(g) \\ &\quad + f(\cdot)((AB)(\cdot))(g)\end{aligned}$$

However the commutator, $[A,B] \equiv AB - BA$, does:

$$\begin{aligned}[A,B](fg) &= (AB)(f)g + \cancel{B(f)A(g)} + \cancel{A(f)B(g)} + f(AB)(g) \\ &\quad - (BA)(f)g - \cancel{A(f)B(g)} - \cancel{B(f)A(g)} - f(BA)(g) \\ &= ([A,B])(f)g + f([A,B])(g) \quad \checkmark\end{aligned}$$

**2** (i) <u>Cl.:</u> $[A,[B,C]] + (\text{cyclic permutations}) = 0$

<u>Pf.:</u> $[A,[B,C]] \equiv [A, BC - CB] = ABC - ACB - BCA + CBA$

$[A,[B,C]] + (\text{cyc. perm.}) \equiv [A,[B,C]] + [B,[C,A]] + [C,[A,B]] =$

$= \cancel{ABC} - \cancel{ACB} - \cancel{BCA} + \cancel{CBA}$
$+ \cancel{BCA} - \cancel{BAC} - \cancel{CAB} + \cancel{ACB}$
$+ \cancel{CAB} - \cancel{CBA} - \cancel{ABC} + \cancel{BAC} = 0$ ✓

(ii) Let $\{Y_i\}_{i=1}^n \subseteq \Gamma(M)$ be $n$-vector fields s.t. $\forall p \in M$, $\{Y_i(p)\}_{i=1}^n$ is a basis of $T_pM$.

Note $[Y_i, Y_j] \in \Gamma(M)$, so we may expand it at each point $p \in M$ using the basis $\{Y_i(p)\}_{i=1}^n$:

$[Y_i, Y_j] =: C^{ij}_k \, Y_k$ (this eq-n defines the expansion coefficients $C^{ij}_k$).

<u>Cl.:</u> $C^{ij}_k = -C^{ji}_k$

<u>Pf.:</u> $C^{ij}_k \equiv [Y_i, Y_j]_k = (-[Y_j, Y_i])_k \overset{\text{Component map is linear}}{=} -[Y_j, Y_i]_k \equiv -C^{ji}_k$

Note that since everything is a function of the point $p \in M$, $C^{ij}_k$ are also $p$-dependent and so they are maps $M \to \mathbb{R}$.

<u>Cl.:</u> $C^{ij}_k C^{\cdot\cdot}_{\cdot} + C_{\cdot} C^{\cdot\cdot}_{\cdot} + C^{\cdot\cdot}_{\cdot} = Y_{\cdot} C^{\cdot\cdot}_{\cdot} + Y_{\cdot} C^{\cdot\cdot}_{\cdot} + Y_{\cdot} C^{\cdot\cdot}_{\cdot}$

<u>Pf.:</u> By the Jacobi identity we have for any $(i,j,k)$ in $\{1,\dots,n\}^3$:

$[Y_i, [Y_j, Y_k]] + (\text{cyclic perm.}) = 0$

$\Leftrightarrow \quad \underbrace{[Y_i, C^{jk}_l Y_l]}$ + cyc. perm of $(i,j,k)$ $= 0$

$$\underbrace{Y_i(C^{jk}_l Y_l)} - C^{jk}_l Y_l Y_i$$

$$Y_i(C^{jk}_l) Y_l + C^{jk}_l Y_i Y_l$$

So $[Y_i, C^{jk}_l Y_l] = Y_i(C^{jk}_l) Y_l + C^{jk}_l [Y_i, Y_l]$

$$= Y_i(C^{jk}_l) Y_l + C^{jk}_l C^{il}_m Y_m$$ relabel $m \leftrightarrow l$

$$= Y_i(C^{jk}_l) Y_l + C^{jk}_m C^{im}_l Y_l$$

$$= \left(Y_i(C^{jk}_l) + C^{jk}_m C^{im}_l\right) Y_l$$

Since $\{Y_l(p)\}_l$ forms a basis, we find

$$\boxed{Y_i(C^{jk}_l) + C^{jk}_m C^{im}_l + \text{cyclic perm. of } (i,j,k) = 0 \quad \forall l}$$

## Q2 About the Lie Derivative

Let $M$ be a smooth manifold of dim $n \in \mathbb{N}_{\geq 1}$.
Let $X \in \Gamma(M)$ (a vector field; a section on $TM$).
That means that there is a flow $\underbrace{\varphi(X) : (-\varepsilon, \varepsilon) \to Aut(M)}_{\text{group morphism}}$ associated
where $((-\varepsilon,\varepsilon),+)$ is the additive g

with $X$ as follows:

$$(*) \begin{cases} \partial_t\left(\cdot \, \mapsto ((\varphi(X))(t))(p)\right) = X(((\varphi(X))(t))(p)) \quad \forall t \in (-\varepsilon,\varepsilon) \\ \varphi(X)(0) = 1_M \end{cases}$$

Note that the L.H.S. of the first equation is a tangent vector at $((\varphi(X))(t))(p)$ indeed. Also note that we

# HW2 Q2

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## 1 Notation

Let $\varphi : U_\varphi \to \mathbb{R}^n$ and $\psi : U_\psi \to \mathbb{R}^n$ be two charts near some $p \in \mathcal{M}$.

Then we define basis vectors of $T_p\mathcal{M}$ corresponding to these charts as $d_i^\varphi := \left[\partial_i \left(\cdot \circ \varphi^{-1}\right)\right] \circ \varphi$. Note that this is really a vector *field* defined in a neighborhoud of $p$. In a point $q \in \mathcal{M}$ it is a tangent vector: $d_i^\varphi$ at $q$ is $\partial_i|_{\varphi(q)} \left(\cdot \circ \varphi^{-1}\right)$. There are analogous definitions for $\psi$. We define the expansion coefficients of a vector field $X$ in the basis corresponding to $\varphi$ as $X_i^\varphi$:

$$X = X_i^\varphi d_i^\varphi$$

so that $X_i^\varphi \equiv X(\varphi_i)$ with $\varphi_i := \pi_i \circ \varphi$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the natural projection. The transition rule (going from $\varphi$ to $\psi$) for the expansion coefficients may be derived easily as

$$\begin{aligned} X_i^\psi &\equiv X\left(d_i^\psi\right) \\ &= X_j^\varphi d_j^\varphi (\psi_i) \end{aligned}$$

so that we define

$$M_{ij}^{\psi\varphi} := d_j^\varphi (\psi_i)$$

and get

$$X_i^\psi = M_{ij}^{\psi\varphi} X_j^\varphi$$

Similarly, we can move the basis vectors themselves:

$$\begin{aligned} d_i^\psi &= d_i^\psi (\varphi_j)\, d_j^\varphi \\ &= M_{ji}^{\varphi\psi} d_j^\varphi \\ &=: N_{ij}^{\psi\varphi} d_j^\varphi \end{aligned}$$

We also have a natural basis for $(T_p\mathcal{M})^*$, given by the dual of $d_i^\varphi$. Explicitly it is given by

$$e_i^\varphi := \cdot(\varphi_i)$$

That is, given any tangent vector $X$, $e_i^\varphi (X) \equiv X(\varphi_i) = X_i^\varphi$. The expansion coefficients of a 1-form $\omega$ are given by

$$\omega_i^\varphi = \omega\left(d_i^\varphi\right)$$

so that

$$\omega = \omega_i^\varphi e_i^\varphi$$

and the transformation rule for the expansion coefficients is

$$\begin{aligned} \omega_i^\psi &\equiv \omega\left(d_i^\psi\right) \\ &= \omega_j^\varphi e_j^\varphi \left(d_i^\psi\right) \end{aligned}$$

But $e_j^\varphi \left(d_i^\psi\right) \equiv d_i^\psi (\varphi_j) = N_{ij}^{\psi\varphi}$ so that we get

$$\omega_i^\psi = N_{ij}^{\psi\varphi} \omega_j^\varphi$$

and of course the dual basis vectors transform again in the opposite way compared to the expansion coefficients:

$$\begin{aligned} e_i^\psi &= e_i^\psi \left(d_j^\varphi\right) e_j^\varphi \\ &= d_j^\varphi (\psi_i)\, e_j^\varphi \\ &= M_{ij}^{\psi\varphi} e_j^\varphi \end{aligned}$$

We find that the expansion coefficients of a general $(k, l)$ tensor $T$ transform as

$$T^\psi_{i_1\cdots i_k j_1 \cdots j_l} = M^{\psi\varphi}_{i_1 i_1'} \cdots M^{\psi\varphi}_{i_k i_k'} N^{\psi\varphi}_{j_1 j_1'} \cdots N^{\psi\varphi}_{j_l j_l'} T^\varphi_{i_1' \cdots i_k' j_1' \cdots j_l'}$$

## 2 Properties of the Transition Matrices

*1 Claim.* We have $N^{\psi\varphi}_{ij} M^{\psi\varphi}_{ik} = \delta_{jk}$ and $N^{\psi\varphi}_{ij} M^{\psi\varphi}_{kj} = \delta_{ik}$.

*Proof.* We start by plugging in the definitions

$$N^{\psi\varphi}_{ij} M^{\psi\varphi}_{ik} \equiv d^{\psi}_{i}(\varphi_j)\, d^{\varphi}_{k}(\psi_i)$$

we swap out $\varphi_j$ and $\psi_i$ for $e^{\varphi}_{j}$ and $e^{\psi}_{i}$ respectively, because it is more transparent then that these are dual vectors to the $d$'s. We get

$$\begin{aligned} N^{\psi\varphi}_{ij} M^{\psi\varphi}_{ik} &= d^{\psi}_{i}\left(e^{\varphi}_{j}\right) d^{\varphi}_{k}\left(e^{\psi}_{i}\right) \\ &= d^{\varphi *}_{j}\left(d^{\psi}_{i}\right) d^{\psi *}_{i}\left(d^{\varphi}_{k}\right) \\ &= \left\langle d^{\varphi}_{j},\, d^{\psi}_{i}\right\rangle \left\langle d^{\psi}_{i},\, d^{\varphi}_{k}\right\rangle \\ &= \left\langle d^{\varphi}_{j},\, d^{\psi}_{i} \otimes d^{\psi *}_{i} d^{\varphi}_{k}\right\rangle \end{aligned}$$

Now we use the fact that $d^{\psi}_{i} \otimes d^{\psi *}_{i} = \mathbb{1}$ because $\left\{\, d^{\psi}_{i} \,\right\}_{i=1}^{n}$ is an ONB of $T_p\mathcal{M}$ for each $p$ in the domain of that basis. Thus

$$N^{\psi\varphi}_{ij} M^{\psi\varphi}_{ik} = \left\langle d^{\varphi}_{j},\, d^{\varphi}_{k}\right\rangle$$

and again using the fact that $\{\, d^{\varphi}_{i} \,\}_{i=1}^{n}$ is a basis one obtains the proper result. The other result is obtained by repeating the argument with $\varphi \leftrightarrow \psi$. □

**2 Corollary.** *We have* $d^{\varphi}_{l}\left(N^{\psi\varphi}_{ij}\right) M^{\psi\varphi}_{ik} = -N^{\psi\varphi}_{ij} d^{\varphi}_{l}\left(M^{\psi\varphi}_{ik}\right)$.

*Proof.* Apply $d^{\varphi}_{l}$ on the foregoing equation. Since $\delta_{ik}$ is a constant scalar function, we get zero on the left hand side (as a tangent vector working on any scalar function is zero). On the right hand side we use the Leibniz property of $d^{\varphi}_{l}$. □

*3 Claim.* We have $d^{\varphi}_{k}\left(M^{\psi\varphi}_{ii'}\right) = d^{\varphi}_{i'}\left(M^{\psi\varphi}_{ik}\right)$.

*Proof.* If we expand out the definitions we will find that this boils down to the fact that $[d^{\varphi}_{i},\, d^{\varphi}_{k}] = 0$, which is always true for basis tangent vectors which correspond to charts, which is what $d^{\varphi}_{i}$ is. Indeed,

$$\begin{aligned} M_{ii',\,k} - M_{ik,\,i'} &\equiv d^{\varphi}_{k}\left(M_{ii'}\right) - d^{\varphi}_{i'}\left(M_{ik}\right) \\ &= d^{\varphi}_{k}\left(d^{\varphi}_{i'}\left(\psi_i\right)\right) - d^{\varphi}_{i'}\left(d^{\varphi}_{k}\left(\psi_i\right)\right) \\ &= [d^{\varphi}_{k},\, d^{\varphi}_{i'}]\left(\psi_i\right) \end{aligned}$$

and $\left[d^{\varphi}_{i},\, d^{\varphi}_{j}\right] = 0$ because

$$\begin{aligned} \left(\left[d^{\varphi}_{i},\, d^{\varphi}_{j}\right]\right)(f) &\equiv d^{\varphi}_{i} d^{\varphi}_{j} f - (i \leftrightarrow j) \\ &= \left[\partial_i \left(d^{\varphi}_{j} f \circ \varphi^{-1}\right)\right] \circ \varphi - (i \leftrightarrow j) \\ &= \left[\partial_i \left(\left[\partial_j \left(f \circ \varphi^{-1}\right)\right] \circ \varphi \circ \varphi^{-1}\right)\right] \circ \varphi - (i \leftrightarrow j) \\ &= \left[\partial_i \left(\partial_j \left(f \circ \varphi^{-1}\right)\right)\right] \circ \varphi - (i \leftrightarrow j) \\ &= 0 \end{aligned}$$

as $\partial_i \partial_j = \partial_j \partial_i$. □

## 3 Some short hand notation to make the calculation lighter

From this point onwards, since the charts $\varphi$ and $\psi$ are fixed, we omit them from the notation. Thus $\varphi$ is considered the "original" chart and $\psi$ the "new" chart. Consequently, all expansion coefficients in the original chart $\varphi$ will have $\varphi$ simply dropped expansion coefficients in the new chart $\psi$ will be denoted by a bar above. We also abbreviate $M^{\psi\varphi}_{ij}$ simply as $M_{ij}$ and the same for $N$. Finally we also abbreviate $d^{\varphi}_{i}(O) \equiv O_{,\,i}$ for any object $O$ (typically $O$ is an expansion coefficient in $\varphi$ or $\psi$ carrying itself some indices, but the application of $d^{\varphi}_{i}$ always will be noted with a comma after all other indices).

Hence the transformation law for a vector's expansion coefficients

$$\overline{X}_i = M_{ij} X_i$$

The transformation law for a dual vector's expansion coefficients

$$\overline{\mu}_i = N_{ij} \mu_i$$

Transformation law for a (1, 1) tensor's expansion coefficients

$$\overline{T}_{ij} = M_{ii'} N_{jj'} T_{i'j'}$$

Transformation law for a basis vector

$$\overline{d}_i = N_{ij} d_j$$

In the exercise, we "define" the Lie derivative along a vector field $X$ of the (1, 1) tensor $T$ via its components as

$$(L_X T)_{ij} = T_{ij,\,k} X_k - T_{kj} X_{i,\,k} + T_{ik} X_{k,\,j}$$

To see how it transforms, we must see how its constituent parts transform:

$$\begin{aligned} \overline{X}_{i,\,j} &\equiv \overline{d}_j \left(\overline{X}_i\right) \\ &= N_{jj'} d_{j'} \left(M_{ii'} X_{i'}\right) \\ &= N_{jj'} d_{j'} \left(M_{ii'}\right) X_{i'} + N_{jj'} M_{ii'} d_{j'} \left(X_{i'}\right) \\ &= N_{jj'} M_{ii',\,j'} X_{i'} + N_{jj'} M_{ii'} X_{i',\,j'} \end{aligned}$$

So that $\overline{X}_{i,\,j}$ does *not* transform like a (1, 1) tensor, due to the extra first term (the second term alone is how it should have transformed had it been a (1, 1) tensor).

We have also

$$\begin{aligned} \overline{T}_{ij,\,k} &\equiv \overline{d}_k \left(\overline{T}_{ij}\right) \\ &= N_{kk'} d_{k'} \left(M_{ii'} N_{jj'} T_{i'j'}\right) \\ &= N_{kk'} M_{ii',\,k'} N_{jj'} T_{i'j'} + N_{kk'} M_{ii'} N_{jj',\,k'} T_{i'j'} + N_{kk'} M_{ii'} N_{jj'} T_{i'j',\,k'} \end{aligned}$$

So that $\overline{T}_{ij,\,k}$ does *not* transform like a (1, 2) tensor, due to the extra first two terms (the third term alone is how a (1, 2) tensor should have transformed).

We check however the transformation law of $\overline{(L_X T)}_{ij}$:

$$\begin{aligned} \overline{(L_X T)}_{ij} &= \overline{T}_{ij,\,k} \overline{X}_k - \overline{T}_{kj} \overline{X}_{i,\,k} + \overline{T}_{ik} \overline{X}_{k,\,j} \\ &= \left(N_{kk'} M_{ii',\,k'} N_{jj'} T_{i'j'} + N_{kk'} M_{ii'} N_{jj',\,k'} T_{i'j'} + N_{kk'} M_{ii'} N_{jj'} T_{i'j',\,k'}\right) M_{kk''} X_{k''} \\ &\quad - M_{kk'} N_{jj'} T_{k'j'} \left(N_{kk''} M_{ii',\,k''} X_{i'} + N_{kk''} M_{ii'} X_{i',\,k''}\right) \\ &\quad + M_{ii'} N_{kk'} T_{i'k'} \left(N_{jj'} M_{kk'',\,j'} X_{k''} + N_{jj'} M_{kk''} X_{k'',\,j'}\right) \\ &\quad \text{(Regroup to terms containing derivatives of N and M and those that don't)} \\ &= N_{kk'} M_{ii',\,k'} N_{jj'} M_{kk''} T_{i'j'} X_{k''} + N_{kk'} M_{ii'} N_{jj',\,k'} M_{kk''} T_{i'j'} X_{k''} \\ &\quad - M_{kk'} N_{jj'} N_{kk''} M_{ii',\,k''} T_{k'j'} X_{i'} + M_{ii'} N_{kk'} N_{jj'} M_{kk'',\,j'} T_{i'k'} X_{k''} \\ &\quad - M_{kk'} N_{jj'} N_{kk''} M_{ii'} T_{k'j'} X_{i',\,k''} \\ &\quad + N_{kk'} M_{ii'} N_{jj'} M_{kk''} T_{i'j',\,k'} X_{k''} + M_{ii'} N_{kk'} N_{jj'} M_{kk''} T_{i'k'} X_{k'',\,j'} \end{aligned}$$

We know what the answer *should* be:

$$\begin{aligned} \overline{(L_X T)}_{ij} &\stackrel{?}{=} M_{ii'} N_{jj'} \left(L_X T\right)_{i'j'} \\ &= M_{ii'} N_{jj'} T_{i'j',\,k} X_k - M_{ii'} N_{jj'} T_{kj'} X_{i',\,k} + M_{ii'} N_{jj'} T_{i'k} X_{k,\,j'} \end{aligned}$$

So we identify those terms in $\overline{(L_X T)}_{ij}$ as $C$ (for "correct", the last two lines) and $R$ (for "rest", the first two lines):

$$\begin{aligned} C &:= N_{kk'} M_{ii'} N_{jj'} M_{kk''} T_{i'j',\,k'} X_{k''} - M_{kk'} N_{jj'} N_{kk''} M_{ii'} T_{k'j'} X_{i',\,k''} \\ &\quad + M_{ii'} N_{kk'} N_{jj'} M_{kk''} T_{i'k'} X_{k'',\,j'} \end{aligned}$$

and

$$\begin{aligned} R &:= N_{kk'} M_{ii',\,k'} N_{jj'} M_{kk''} T_{i'j'} X_{k''} + N_{kk'} M_{ii'} N_{jj',\,k'} M_{kk''} T_{i'j'} X_{k''} \\ &\quad - M_{kk'} N_{jj'} N_{kk''} M_{ii',\,k''} T_{k'j'} X_{i'} + M_{ii'} N_{kk'} N_{jj'} M_{kk'',\,j'} T_{i'k'} X_{k''} \end{aligned}$$

We want to show that

$$C \overset{?}{=} M_{ii'} N_{jj'} T_{i'j',k} X_k - M_{ii'} N_{jj'} T_{kj'} X_{i',k} + M_{ii'} N_{jj'} T_{i'k} X_{k,j'} \quad (1)$$

and that $R = 0$.

We start with the first task. In order to do that we must we must "cancel out" factors of $M$ and $N$. Take for instance the first term in $C$:

$$N_{kk'} M_{ii'} N_{jj'} M_{kk''} T_{i'j',k'} X_{k''} = (M_{ii'} N_{jj'} T_{i'j',k'}) (N_{kk'} M_{kk''} X_{k''})$$

Using 1 we find for that term

$$\begin{aligned} N_{kk'} M_{ii'} N_{jj'} M_{kk''} T_{i'j',k'} X_{k''} &= (M_{ii'} N_{jj'} T_{i'j',k'}) (N_{kk'} M_{kk''} X_{k''}) \\ &= (M_{ii'} N_{jj'} T_{i'j',k'}) (\delta_{k'k''} X_{k''}) \\ &= (M_{ii'} N_{jj'} T_{i'j',k'}) X_{k'} \end{aligned}$$

so we get the first term on the RHS of (1) correctly. We proceed similarly using 1 twice more to find that (1) is correct.

We go on to prove that $R = 0$: We use 1 three more times to find:

$$\begin{aligned} R \equiv\ & N_{kk'} M_{ii',k'} N_{jj'} M_{kk''} T_{i'j'} X_{k''} + N_{kk'} M_{ii'} N_{jj',k'} M_{kk''} T_{i'j'} X_{k''} \\ & - M_{kk'} N_{jj'} N_{kk''} M_{ii',k''} T_{k'j'} X_{i'} + M_{ii'} N_{kk'} N_{jj'} M_{kk'',j'} T_{i'k'} X_{k''} \\ =\ & M_{ii',k} N_{jj'} T_{i'j'} X_k - N_{jj'} M_{ii',k} T_{kj'} X_{i'} \\ & + M_{ii'} N_{jj',k} T_{i'j'} X_k + M_{ii'} N_{kk'} N_{jj'} M_{kk'',j'} T_{i'k'} X_{k''} \end{aligned}$$

for the first line, in the second term we relabel as $i' \leftrightarrow k$ to get

$$M_{ii',k} N_{jj'} T_{i'j'} X_k - N_{jj'} M_{ii',k} T_{kj'} X_{i'} = M_{ii',k} N_{jj'} T_{i'j'} X_k - N_{jj'} M_{ik,i'} T_{i'j'} X_k$$

But now use 3 so the first line of our most recent expression for $R$ is zero.

We go on to the next line. We relabel in the second term $j' \leftrightarrow k'$ and $k \leftrightarrow k''$ to get

$$M_{ii'} N_{jj',k} T_{i'j'} X_k + M_{ii'} N_{kk'} N_{jj'} M_{kk'',j'} T_{i'k'} X_{k''} = M_{ii'} (N_{jj',k} + N_{k''j'} N_{jk'} M_{k''k,k'}) T_{i'j'} X_k$$

Now we deal with the term $N_{k''j'} N_{jk'} M_{k''k,k'}$. In fact we can rewrite it as

$$\begin{aligned} N_{k''j'} N_{jk'} M_{k''k,k'} &= N_{k''j'} N_{jk'} M_{k''k',k} \\ &= -N_{k''j',k} N_{jk'} M_{k''k'} \\ &= -N_{k''j',k} \delta_{jk''} \\ &= -N_{jj',k} \end{aligned}$$

so that we really get zero. In the last expression, used again the fact that $M_{ij,k} = M_{ik,j}$ (in 3) as proven above already, as well as 2. The proof is finally complete.