The Jacobian Identity

(Wald Ch. 2 Ex. 3)

Let $A, B, C$ be vector fields on the manifold $M$. That means they are sections in the tangent bundle $TM$.

Let $f \in \mathcal{F}(M)$. For any point $p \in TM$, so $A(p) : \mathcal{F}(M) \to \mathbb{R}$. Hence $(A(p))(p) \in \mathbb{R}$. Thus $p \mapsto (A(p))(p)$ is again in $\mathcal{F}(M)$.

Thus, it makes sense to compose vector fields together, as $A \circ B = A(p \mapsto (B(p))(\cdot))$. [A acts on the map where]

However, note that $A \circ B$ is not a tangent field since it doesn't have the Leibnitz property:

$$(AB)(fg) = A(p \mapsto (B(p))(f(g)))$$

But Leibnitz -

$= A(p \mapsto (B(p))(f(g)) + f(p))(B(p))(g))$$

$= A(p \mapsto (B(p))(f(g)) + (f(p))(B(p))(g))$$

$= A(p \mapsto f(p))(B(p))(g)) + A(p \mapsto B(p))(g))$$

$= (A(\cdot))(p) f(g) + B(\cdot)(p) A(\cdot)(g) + (A(\cdot))(p) B(\cdot)(g)$$

However, the commutator $[A, B] = AB - BA$, does:

$[AB](fg) = (AB)(p) f(g) + B(p)(A(g)) + A(p)(B(g))$$

$= (BA)(p) g - A(p)(B(g)) - B(p)(A(g))$$

$= (LA, B)(p) g + p(LA, B)(g)$
\[ \text{(i) } \mathcal{C}_{i}^{0} \quad [A, [B, C]] + (\text{cyclic permutations}) = 0 \]

\[ \text{Pr.1.} \quad [A, [B, C]] = [A, BC - CB] = ABC - ACB - BCA + CBA \]


\[ = ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC = 0 \]

\[ \text{(ii)} \quad \text{Let } \{Y_{i}\}_{i=1}^{n} \subseteq \Gamma(M) \text{ be } n \text{-vector fields s.t. } V_{p}M. \]

\[ \{Y_{i}(p)\}_{i=1}^{n} \text{ is a basis of } T_{p}M. \]

\[ \text{Note } [Y_{i}, Y_{j}] \in \Gamma(M), \text{ so we may expand it at each point } p \in M \text{ using the basis } \{Y_{i}(p)\}_{i=1}^{n}: \]

\[ [Y_{i}, Y_{j}] = C_{k}^{i} Y_{k} \quad \text{(this eqn defines the expansion coefficients } C_{k}^{i}). \]

\[ \text{Cl.1.} \quad C_{k}^{i} = -C_{k}^{i} \]

\[ \text{Pr.1.} \quad C_{k}^{i} = [Y_{i}, Y_{j}]_{k} = (-[Y_{j}, Y_{i}])_{k} = -[Y_{j}, Y_{i}]_{k} \]

\[ = -C_{k}^{i} \]

\[ \text{Note that since everything is a function of the point } p \in M, \]
\[ C_{k}^{i} \text{ are also } p \text{-dependent and so they are maps } M \rightarrow \mathbb{R}. \]

\[ \text{Cl.1.} \quad C_{k}^{i} C_{j}^{k} + C_{k}^{i} C_{i}^{j} = Y.C_{i} + Y.C_{j} + Y.C^{i} + Y.C^{j} \]

\[ \text{Pr.2.} \quad \text{By the Jacobi identity we have for any } (i, j, k) \]
\[ \text{in } \{1, \ldots, n\}^3: \]

\[ [Y_{i}, [Y_{j}, Y_{k}]] + (\text{cyclic perm.}) = 0 \]
About the Lie Derivative $\mathcal{L}$

Let $M$ be a smooth manifold of dimension $n \in \mathbb{N}_{\geq 1}$.

Let $X \in \mathfrak{X}(M)$ (a vector field; a section on $TM$).

That means that there is a flow $\phi^X((\cdot),t) : \text{Aut}(M) \to \text{Aut}(M)$ associated with $X$ as follows:

$$\phi^X((\cdot),0) = \text{id}$$

$$\phi^X((\cdot),t) \circ \phi^X((\cdot),s) = \phi^X((\cdot),t+s), \quad \forall t,s \in \mathbb{R}$$

Note that the RHS of the first equation is a tangent vector at $((\phi^X((\cdot)),(t))(p))$ indeed. Also note that we
1 Notation

Let $\varphi : U_\varphi \to \mathbb{R}^n$ and $\psi : U_\psi \to \mathbb{R}^n$ be two charts near some $p \in \mathcal{M}$.

Then we define basis vectors of $T_p \mathcal{M}$ corresponding to these charts as $d_i^\varphi := \left[ \partial_i (\cdot \circ \varphi^{-1}) \right] \circ \varphi$. Note that this is really a vector field defined in a neighborhood of $p$. In a point $q \in \mathcal{M}$ it is a tangent vector: $d_i^\varphi$ at $q$ is $\partial_i|_{\varphi(q)} (\cdot \circ \varphi^{-1})$. There are analogous definitions for $\psi$. We define the expansion coefficients of a vector field $X$ in the basis corresponding to $\varphi$ as $X_i^\varphi$:

$$X = X_i^\varphi d_i^\varphi$$

so that $X_i^\varphi \equiv X (\varphi_i)$ with $\varphi_i := \pi \circ \varphi$ and $\pi : \mathbb{R}^n \to \mathbb{R}$ is the natural projection. The transition rule (going from $\varphi$ to $\psi$) for the expansion coefficients may be derived easily as

$$X_i^\psi \equiv X \left( d_i^\psi \right) = X_j^\varphi d_j^\varphi (\psi_i)$$

so that we define

$$M_{ij}^{\psi \varphi} := d_j^\varphi (\psi_i)$$

and get

$$X_i^\psi = M_{ij}^{\psi \varphi} X_j^\varphi$$

Similarly, we can move the basis vectors themselves:

$$d_i^\psi = d_i^\varphi (\varphi_j) d_j^\psi = M_{ij}^{\psi \varphi} d_j^\varphi$$

We also have a natural basis for $(T_p \mathcal{M})^*$, given by the dual of $d_i^\varphi$. Explicitly it is given by

$$e_i^\varphi := \cdot (\varphi_i)$$

That is, given any tangent vector $X$, $e_i^\varphi (X) \equiv X (\varphi_i) = X_i^\varphi$. The expansion coefficients of a 1-form $\omega$ are given by

$$\omega_i^\varphi = \omega (d_i^\varphi)$$

so that

$$\omega = \omega^\varphi e_i^\varphi$$

and the transformation rule for the expansion coefficients is

$$\omega_i^\psi = \omega_i^\varphi (\varphi_j) = N_{ij}^{\psi \varphi} \omega_j^\varphi$$

But $e_j^\psi \left( d_i^\psi \right) \equiv d_i^\psi (\varphi_j) = N_{ij}^{\psi \varphi}$ so that we get

$$\omega_i^\psi = N_{ij}^{\psi \varphi} \omega_j^\varphi$$

and of course the dual basis vectors transform again in the opposite way compared to the expansion coefficients:

$$e_i^\varphi = e_i^\psi (d_i^\varphi) e_j^\varphi = d_j^\varphi (\psi_i) e_j^\varphi = M_{ij}^{\varphi \psi} e_j^\varphi$$

We find that the expansion coefficients of a general $(k, l)$ tensor $T$ transform as

$$T_{i_1 \cdots i_k j_1 \cdots j_l} = M_{i_1 i_k}^{\varphi \psi} M_{j_1 j_l}^{\psi \varphi} N_{i_1}^{\psi \varphi} N_{j_1}^{\psi \varphi} T_{i_k j_l}^{\psi \varphi}$$
2 Properties of the Transition Matrices

1 Claim. We have $N_{ij}^{\psi \varphi} M_{ik}^{\psi \varphi} = \delta_{jk}$ and $N_{ij}^{\psi \varphi} M_{kj}^{\psi \varphi} = \delta_{ik}$.

Proof. We start by plugging in the definitions

$$N_{ij}^{\psi \varphi} M_{ik}^{\psi \varphi} \equiv \frac{\partial}{\partial \phi_j} (\varphi_j) \frac{\partial}{\partial \psi_i} (\psi_i)$$

we swap out $\varphi_j$ and $\psi_i$ for $e_j^\varphi$ and $e_i^\psi$ respectively, because it is more transparent then that these are dual vectors to the $d^i$’s. We get

$$N_{ij}^{\psi \varphi} M_{ik}^{\psi \varphi} = d_i^\psi (e_j^\varphi) d_k^\psi (e_i^\psi)$$

Now we use the fact that $d_i^\psi \otimes d_i^{\psi *} = \mathbf{1}$ because $\{d_i^\psi\}_{i=1}^n$ is an ONB of $T_pM$ for each $p$ in the domain of that basis. Thus

$$N_{ij}^{\psi \varphi} M_{ik}^{\psi \varphi} = \langle d_i^\psi, d_k^\psi \rangle$$

and again using the fact that $\{d_i^\psi\}_{i=1}^n$ is a basis one obtains the proper result. The other result is obtained by repeating the argument with $\varphi \leftrightarrow \psi$.  

2 Corollary. We have $d_i^\psi \left( N_{ij}^{\psi \varphi} \right) M_{ik}^{\psi \varphi} = -N_{ij}^{\psi \varphi} d_i^\varphi \left( M_{ik}^{\psi \varphi} \right)$.

Proof. Apply $d_i^\psi$ on the foregoing equation. Since $\delta_{ik}$ is a constant scalar function, we get zero on the left hand side (as a tangent vector working on any scalar function is zero). On the right hand side we use the Leibniz property of $d_i^\varphi$.  

3 Claim. We have $d_k^\psi \left( M_{ij}^{\psi \varphi} \right) = d_i^\varphi \left( M_{ik}^{\psi \varphi} \right)$.

Proof. If we expand out the definitions we will find that this boils down to the fact that $[d_i^\varphi, d_j^\varphi] = 0$, which is always true for basis tangent vectors which correspond to charts, which is what $d_i^\varphi$ is. Indeed,

$$M_{ii', k} - M_{ik, i'} \equiv d_k^\psi (M_{ii'}) - d_k^\psi (M_{ik})$$

$$= d_k^\psi (d_i^\varphi (\psi_i)) - d_k^\psi (d_i^\psi (\psi_i))$$

$$= [d_k^\psi, d_i^\varphi] (\psi_i)$$

and $[d_i^\varphi, d_j^\varphi] = 0$ because

$$\left( [d_i^\varphi, d_j^\varphi] (f) \right) = d_i^\varphi d_j^\varphi f - (i \leftrightarrow j)$$

$$= [\partial_i (d_j^\varphi f \circ \varphi^{-1})] \circ \varphi - (i \leftrightarrow j)$$

$$= [\partial_i (d_j (f \circ \varphi^{-1})) \circ \varphi \circ \varphi^{-1})] \circ \varphi - (i \leftrightarrow j)$$

$$= [\partial_i (d_j (f \circ \varphi^{-1}))] \circ \varphi - (i \leftrightarrow j)$$

$$= 0$$

as $\partial_i \partial_j = \partial_j \partial_i$.

3 Some short hand notation to make the calculation lighter

From this point onwards, since the charts $\varphi$ and $\psi$ are fixed, we omit them from the notation. Thus $\varphi$ is considered the “original” chart and $\psi$ the “new” chart. Consequently, all expansion coefficients in the original chart $\varphi$ will have $\varphi$ simply dropped expansion coefficients in the new chart $\psi$ will be denoted by a bar above. We also abbreviate $M_{ij}^{\psi \varphi}$ simply as $M_{ij}$ and the same for $N$. Finally we also abbreviate $d_i^\varphi (O) \equiv O_i$ for any object $O$ (typically $O$ is an expansion coefficient in $\varphi$ or $\psi$ carrying itself some indices, but the application of $d_i^\varphi$ always will be noted with a comma after all other indices).
Hence the transformation law for a vector’s expansion coefficients
\[ \overline{X}_i = M_{ij} X_j \]

The transformation law for a dual vector’s expansion coefficients
\[ \overline{\mu}_i = N_{ij} \mu_j \]

Transformation law for a \((1, 1)\) tensor’s expansion coefficients
\[ \overline{T}_{ij} = M_{i\nu} N_{j\nu'} T_{\nu'j'} \]

Transformation law for a basis vector
\[ \overline{d}_i = N_{ij} d_j \]

In the exercise, we “define” the Lie derivative along a vector field \(X\) of the \((1, 1)\) tensor \(T\) via its components as
\[ (L_X T)_{ij} = T_{ij,k} X_k - T_{kj,i} X_{i,k} + T_{ik} X_{k,j} \]

To see how it transforms, we must see how its constituent parts transform:
\[ \overline{X}_{i,j} = \overline{d}_j (\overline{X}_i) = N_{jj'} d_{j'} (M_{i\nu'} X_{\nu'}) = N_{jj'} d_{j'} (M_{i\nu'} X_{\nu'}) + N_{jj'} M_{i\nu'} d_{j'} (X_{\nu'}) \]

So that \(\overline{X}_{i,j}\) does not transform like a \((1, 1)\) tensor, due to the extra first term (the second term alone is how it should have transformed had it been a \((1, 1)\) tensor).

We have also
\[ T_{ij,k} = \overline{d}_k (\overline{T}_{ij}) \]

\[ = N_{kk'} d_{k'} (M_{i\nu'} N_{j\nu'} T_{\nu'j'}) = N_{kk'} M_{i\nu'} N_{j\nu'} T_{\nu'j' k'} + N_{kk'} M_{i\nu'} N_{j\nu'} T_{\nu'j' k'} (\text{for “rest”, the first two lines}) \]

So that \(T_{ij,k}\) does not transform like a \((1, 2)\) tensor, due to the extra first two terms (the third term alone is how a \((1, 2)\) tensor should have transformed).

We check however the transformation law of \((L_X T)_{ij}\):
\[ (L_X T)_{ij} = T_{ij,k} X_k - T_{kj,i} X_{i,k} + T_{ik} X_{k,j} \]

We know the answer should be:
\[ (L_X T)_{ij} \overset{?}{=} M_{i\nu'} N_{j\nu'} (L_X T)_{\nu'j'} \]

So we identify those terms in \((L_X T)_{ij}\) as \(C\) (for “correct”, the last two lines) and \(R\) (for “rest”, the first two lines):
\[ C := M_{i\nu'} N_{j\nu'} M_{kk''} T_{\nu'j'} X_{k''} - M_{kk''} N_{j\nu'} M_{i\nu'} T_{k''j'} X_{\nu''} + M_{i\nu'} N_{kk''} M_{j\nu'} T_{\nu''j'} X_{k''} + M_{i\nu'} N_{kk''} M_{j\nu'} T_{\nu''j'} X_{k''} \]

and
\[ R := M_{kk''} N_{j\nu'} M_{i\nu'} T_{k''j'} X_{\nu''} + M_{kk''} N_{j\nu'} M_{i\nu'} T_{k''j'} X_{\nu''} - M_{kk''} N_{j\nu'} M_{i\nu'} T_{k''j'} X_{\nu''} + M_{i\nu'} N_{kk''} M_{j\nu'} T_{\nu''j'} X_{k''} \]
We want to show that

\[ C \overset{2}{=} M_{ii'}N_{jj'}T_{i'j',k}X_k - M_{ii'}N_{jj'}T_{kj'}X_{i',k} + M_{ii'}N_{jj'}T_{i'k}X_{k,j'} \]  

(1)

and that \( R = 0 \).

We start with the first task. In order to do that we must we must “cancel out” factors of \( M \) and \( N \). Take for instance the first term in \( C \):

\[ N_{kk'}M_{ii'}N_{jj'}M_{kk''}T_{i'j',k}X_{k''} = (M_{ii'}N_{jj'}T_{i'j',k'}) (N_{kk'}M_{kk''}X_{k''}) \]

Using 1 we find for that term

\[ N_{kk'}M_{ii'}N_{jj'}M_{kk''}T_{i'j',k}X_{k''} = (M_{ii'}N_{jj'}T_{i'j',k'}) (N_{kk'}M_{kk''}X_{k''}) \]

\[ = (M_{ii'}N_{jj'}T_{i'j',k'}) (\delta_{k''}X_{k''}) \]

\[ = (M_{ii'}N_{jj'}T_{i'j',k'}) X_{k''} \]

so we get the first term on the RHS of (1) correctly. We proceed similarly using 1 twice more to find that (1) is correct.

We go on to prove that \( R = 0 \): We use 1 three more times to find:

\[ R = N_{kk'}M_{ii',k'}N_{jj'}M_{kk''}T_{i'j',k}X_{k''} + N_{kk'}M_{ii'}N_{jj'}T_{i'j',k'}X_{k''} \]

\[ - M_{kk'}N_{jj'}N_{kk''}M_{ii',k'}T_{i'j',k'}X_{i'} + M_{ii'}N_{kk'}N_{jj'}M_{kk''}T_{i'k'}X_{k''} \]

\[ = M_{ii',k}N_{jj'}T_{i'j',k}X_k - N_{jj'}M_{ii',k}T_{i'j',k'}X_{i'} \]

\[ + M_{ii'}N_{jj'}T_{i'j',k}X_k + M_{ii'}N_{kk'}N_{jj'}M_{kk''}T_{i'k'}X_{k''} \]

for the first line, in the second term we relabel as \( i' \leftrightarrow k \) to get

\[ M_{ii',k}N_{jj'}T_{i'j',k}X_k - N_{jj'}M_{ii',k}T_{i'j',k'}X_{i'} = M_{ii',k}N_{jj'}T_{i'j',k}X_k - N_{jj'}M_{ik',k}T_{i'j',k}X_{k} \]

But now use 3 so the first line of our most recent expression for \( R \) is zero.

We go on to the next line. We relabel in the second term \( j' \leftrightarrow k' \) and \( k \leftrightarrow k'' \) to get

\[ M_{ii'}N_{jj',k}T_{i'j',k}X_k + M_{ii'}N_{kk'}N_{jj'}M_{kk''}T_{i'k'}X_{k''} = M_{ii'} (N_{jj',k} + N_{kk'}N_{jj'}M_{kk''}k',k) T_{i'j',k}X_k \]

Now we deal with the term \( N_{k''j'}N_{jk'}M_{kk''}k,k' \). In fact we can rewrite it as

\[ N_{k''j'}N_{jk'}M_{kk''}k,k' = N_{k''j'}N_{jk'}M_{kk''}k,k \]

\[ = -N_{k''j'}N_{jk'}M_{kk''}k,k \]

\[ = -N_{k''j'}N_{jk'}\delta_{jk'} \]

\[ = -N_{jj'}k \]

so that we really get zero. In the last expression, used again the fact that \( M_{ij,k} = M_{ik,j} \) (in 3) as proven above already, as well as 2. The proof is finally complete.