GR — HW #3 — 5/10/2017

Q1 The Christoffel Symbols in Polar Coord.

$\mathcal{M} = \mathbb{R}^2$, $\varphi : \mathcal{M} \to \mathbb{R}^2$ chart with $\varphi = \mathbb{1}_{\mathcal{M}}$

$\psi : \mathcal{U} \to \psi(\mathcal{U})$ polar chart; $\psi(x) := \begin{bmatrix} \|x\| \\ \operatorname{arctg}\left(\frac{x_2}{x_1}\right) \end{bmatrix}$

$\mathcal{U} = \mathbb{R}^2 \setminus \{0\}$

$$M^{\psi\varphi}_{ij} \equiv d^{\varphi}_j(\psi_i) \equiv \left[\partial_j(\psi_i \circ \varphi^{-1})\right]\Big|_{\varphi} \overset{\varphi=\mathbb{1}}{=} \partial_j \psi_i$$

$$M^{\psi\varphi}_{1j} = \partial_j \psi_1 = \partial_j \|x\| = \|x\|^{-1} x_j$$

$$M^{\psi\varphi}_{2j} = \partial_j \psi_2 = \partial_j \operatorname{arctg}\left(\frac{x_2}{x_1}\right) = \underbrace{\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}}_{\frac{x_1^2}{\|x\|^2}} \partial_j\left(\frac{x_2}{x_1}\right)$$

$$M^{\psi\varphi}_{21} = \frac{x_1^2}{\|x\|^2}\partial_1\left(\frac{x_2}{x_1}\right) = \frac{\cancel{x_1^2}\, x_2}{\|x\|^2}(-1)\frac{1}{\cancel{x_1^2}} = -\frac{x_2}{\|x\|^2}$$

$$M^{\psi\varphi}_{22} = \frac{x_1^2}{\|x\|^2}\partial_2\left(\frac{x_2}{x_1}\right) = \frac{x_1}{\|x\|^2}$$

$$\Rightarrow \quad M^{\psi\varphi}(x) = \|x\|^{-1}\begin{bmatrix} x_1 & x_2 \\ -\frac{x_2}{\|x\|} & \frac{x_1}{\|x\|} \end{bmatrix} \quad \forall\, x \in \mathbb{R}^2$$

Note $\quad (M^{\psi\varphi} \circ \psi^{-1})(r,\varphi) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r^{-1}\sin(\varphi) & r^{-1}\cos(\varphi) \end{bmatrix}$

$$M^{\varphi\psi}_{ij} \equiv d^{\psi}_j(\varphi_i) = \left[\partial_j(\varphi_i \circ \psi^{-1})\right]\Big|_{\psi} \overset{\varphi=\mathbb{1}}{=} \left[\partial_j(\pi_i \circ \psi^{-1})\right]\Big|_{\psi}$$

$\psi^{-1}(x) = \begin{bmatrix} x_1\cos(x_2) \\ x_1\sin(x_2) \end{bmatrix}$ may be verified.

$$\Rightarrow \left.\begin{aligned} M^{\varphi\psi}_{11} &= \partial_1\, x_1\cos(x_2) = \cos(x_2) \\ M^{\varphi\psi}_{12} &= \partial_2\, x_1\cos(x_2) = -x_1\sin(x_2) \\ M^{\varphi\psi}_{21} &= \partial_1\, x_1\sin(x_2) = \sin(x_2) \\ M^{\varphi\psi}_{22} &= \partial_2\, x_1\sin(x_2) = x_1\cos(x_2) \end{aligned}\right\} \text{ evaluated at } \psi^{-1}(x),$$

$$\Rightarrow \quad M^{\varphi\psi}(x) = \begin{bmatrix} \frac{x_1}{\|x\|} & -x_2 \\ \frac{x_2}{\|x\|} & x_1 \end{bmatrix} \quad \forall\, x \in \mathbb{R}^2.$$

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$$N^{\psi\varphi}_{i\,j} \equiv M^{\varphi\psi}_{j\,i} \qquad \text{(inverse \& transpose)}$$

We know by (2.5) in the lecture notes that:

$$(\Gamma)^{\psi}_{i\,;jk} = M^{\psi\varphi}_{ii'} N^{\psi\varphi}_{jj'} N^{\psi\varphi}_{kk'} (\Gamma)^{\varphi}_{i'\,;j'k'} + M^{\psi\varphi}_{ii'}\, d^{\psi}_{k}(\underbrace{N^{\psi\varphi}_{j\,i'}}_{\text{transposes } N})$$

By definition $(\Gamma)^{\varphi} = 0$. (this is the input of the exer.)

$$\Rightarrow (\Gamma)^{\psi}_{i;jk} = M^{\psi\varphi}_{i\ell}\, d^{\psi}_{k}(N^{\psi\varphi}_{\ell j})$$

For convenience we write everything with ψ^{-1}.

$$(M^{\psi\varphi}\circ\psi^{-1})(r,\varphi) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r^{-1}\sin(\varphi) & r^{-1}\cos(\varphi) \end{bmatrix} \quad \checkmark$$

$$(N^{\psi\varphi}\circ\psi^{-1})(r,\varphi) = ((M^{\varphi\psi})^{T}\circ\psi^{-1})(r,\varphi) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r\sin(\varphi) & r\cos(\varphi) \end{bmatrix} \quad \checkmark$$

$$\Rightarrow d^{\psi}_{r}(N^{\psi\varphi\,T}) = \begin{bmatrix} 0 & -\sin(\varphi) \\ 0 & \cos(\varphi) \end{bmatrix} \qquad d^{\psi}_{\varphi}(N^{\psi\varphi T}) = \begin{bmatrix} -\sin(\varphi) & -r\cos(\varphi) \\ \cos(\varphi) & -r\sin(\varphi) \end{bmatrix}$$

$$(\Gamma)^{\psi}_{\cdot\,;\cdot\,r} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r^{-1}\sin(\varphi) & r^{-1}\cos(\varphi) \end{bmatrix}\begin{bmatrix} 0 & -\sin(\varphi) \\ 0 & \cos(\varphi) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & r^{-1} \end{bmatrix} \quad \checkmark$$

$$(\Gamma)^{\psi}_{\cdot\,;\cdot\,\varphi} = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r^{-1}\sin(\varphi) & r^{-1}\cos(\varphi) \end{bmatrix}\begin{bmatrix} -\sin(\varphi) & -r\cos(\varphi) \\ \cos(\varphi) & -r\sin(\varphi) \end{bmatrix} = \begin{bmatrix} 0 & -r \\ r^{-1} & 0 \end{bmatrix} \quad \checkmark$$

A 2nd Look @ Parallel Transport

a) Tangent bundle $T\mathcal{M} \equiv \bigsqcup_{p\in\mathcal{M}} T_p\mathcal{M}$ (Its points are tuples (q,u), $q\in\mathcal{M}$, $u\in T_q\mathcal{M}$.)

$\pi : T\mathcal{M} \longrightarrow \mathcal{M}$ ↳ proj.
$(p,v) \longmapsto p$

$T\mathcal{M}$ is itself a smooth manifold if $\mathcal{M}$ is:

Let $\varphi : \mathcal{U} \to \mathbb{R}^n$ be a chart at $p\in\mathcal{M}$, $\mathcal{U}\in \mathrm{Open}(\mathcal{M})$.

Note that since π is cont., $\pi^{-1}(\mathcal{U}) \in \mathrm{Open}(T\mathcal{M})$.

Define a chart $\hat{\varphi} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathbb{R}^{2n}$ as

$$(q,u) \longmapsto (\underbrace{\varphi(q)}_{\in\mathbb{R}^n}, \underbrace{u(\varphi_1),\dots,u(\varphi_n)}_{\in\mathbb{R}^n})$$

where $\varphi_i : \mathcal{U}\to\mathbb{R}$ is def. as $\varphi_i := \pi_i\circ\varphi$, $\pi_i : \mathbb{R}^n\to\mathbb{R}$ being the proj. onto the i^{th} coordinate. Recall $u(\varphi_i)$ gives the i^{th} expan. coeff. of $u\in T_q\mathcal{M}$ in the basis of $T_q\mathcal{M}$ induced by the chart φ.

Cl.: With the above procedure for the induced atlas, $T\mathcal{M}$ becomes a smooth manifold.

i) Let $\psi : V\longrightarrow\mathbb{R}^n$ be another chart on $\mathcal{M}$. Then assuming $\mathcal{U}\cap V\neq\emptyset$, $\psi\circ\varphi^{-1} : \varphi(\mathcal{U}\cap V)\to\psi(\mathcal{U}\cap V)$ is a transition map.
We want to calculate $\hat{\psi}\circ\hat{\varphi}^{-1} : \hat{\varphi}(\pi^{-1}(\mathcal{U})\cap\pi^{-1}(V)) \longrightarrow \hat{\psi}(\pi^{-1}(\mathcal{U})\cap\pi^{-1}(V))$
in terms of $\psi\circ\varphi^{-1}$.
Recall that expansion coeff. of vectors in the basis of $T_q\mathcal{M}$ corres. to φ or ψ transformed as

$$u_i^\psi = M^{\psi\varphi}_{ij}\, u_j^\varphi$$

with $M^{\psi\varphi}_{ij} \equiv d^\varphi_j(\psi_i)$ (ψ_i as above, $d^\varphi_i \equiv [\partial_i(\cdot\circ\varphi^{-1})]\circ\varphi$)

Then let $x\in\mathbb{R}^{2n}$ with $x=(a,b)$, $a\in\mathbb{R}^n$, $b\in\mathbb{R}^n$.
Assume $x\in\hat{\varphi}(\pi^{-1}(\mathcal{U})\cap\pi^{-1}(V))$.

$$(\hat{\psi}\circ\hat{\varphi}^{-1})(x) = \hat{\psi}\Big(\big(\underset{\substack{\cap\\ \mathcal{U}\cap V}}{\varphi^{-1}(a)}, \underbrace{\sum_{j=1}^n b_j\, d^\varphi_j(\varphi^{-1}(a))}_{\substack{\cap\\ T_{\varphi^{-1}(a)}\mathcal{M}}}\big)\Big) = \Big((\psi\circ\varphi^{-1})(a), \sum_{j=1}^n b_j d^\varphi_j(\psi_1), \dots, \sum_{j=1}^n b_j d^\varphi_j(\psi_n)\Big)$$

@ $\varphi^{-1}(a)$

$$= \big((\psi\circ\varphi^{-1})(a), (M^{\psi\varphi}b)_1, \dots, (M^{\psi\varphi}b)_n\big)$$

@ $\varphi^{-1}(a)$

$$= \left((\psi\circ\varphi^{-1})(a),\ \underbrace{M^{\psi\varphi}(\varphi^{-1}(a))}\, b \right)$$

$M^{\psi\varphi}$ is a linear map so $\nexists$ need for parenthesis

With, $\pi_L : \mathbb{R}^{2n} \to \mathbb{R}^n$ $(a,b)\mapsto a$ $\pi_R : \mathbb{R}^{2n}\to\mathbb{R}^n$ $(a,b)\mapsto b$, $\mathcal{A} : \mathrm{Mat}_{n\times n}(\mathbb{R})\times\mathbb{R}^n \to \mathbb{R}^n$ $(M,v)\mapsto Mv$ and the convention $f:X\to Y$, $g:X\to Z$ $f\times g : X\to Y\times Z$, $f:X\to Y$, $g:Z\to W$ $(f,g): X\times Z\to Y\times W$:

$$\boxed{\hat\psi\circ\hat\varphi^{-1} = (\psi\circ\varphi^{-1}\circ\pi_L)\times\left(\mathcal{A}\circ(M^{\psi\varphi}\circ\varphi^{-1},\ \mathbb{1})\right)}$$

Next we'd like to compute $M^{\hat\psi\hat\varphi}\in \mathrm{Mat}_{2n\times 2n}(\mathbb{R})$.

$$M^{\hat\psi\hat\varphi}_{ij} \equiv d_j^{\hat\varphi}(\hat\psi_i)$$

We proceed by dividing into cases based on whether i or j ar in $\underbrace{\{1,\dots,n\}}_{I}$ or in $\underbrace{\{n+1,\dots,2n\}}_{II}$

Case 1: $(i,j)\in I^2$

$$\hat\psi_i\circ\hat\varphi^{-1} \equiv \pi_i\circ\hat\psi\circ\hat\varphi^{-1} \overset{\text{by the box above, } i\in I}{=} \pi_i\circ\psi\circ\varphi^{-1}\circ\pi_L = \psi_i\circ\varphi^{-1}\circ\pi_L$$

$$d_j^{\hat\varphi}(\hat\psi_i) \equiv \left[\partial_j(\hat\psi_i\circ\hat\varphi^{-1})\right]\circ\hat\varphi = \left[\partial_j(\psi_i\circ\varphi^{-1}\circ\pi_L)\right]\circ\hat\varphi$$

$$\left(\left\{[\partial_k(\psi_i\circ\varphi^{-1})]\circ\pi_L\right\}\ \partial_j \underbrace{\underbrace{(\pi_L)_k}_{\pi_k}}_{=\delta_{jk}}\right)\circ\hat\varphi = \left[\partial_j(\psi_i\circ\varphi^{-1})\right]\circ\underbrace{\pi_L\circ\hat\varphi}_{\varphi\circ\pi}$$

$$= d_j^{\varphi}(\psi_i)\circ\pi \equiv M^{\psi\varphi}\circ\pi$$

Case 2: $i\in I$ $j\in II$

In this case, $\partial_j(\pi_L)_k = 0$ always as $j>n$.

Case 3: $i\in II$, $j\in I$

$$\hat\psi_i\circ\hat\varphi^{-1} \equiv \pi_i\circ\hat\psi\circ\hat\varphi^{-1} = \pi_{i-n}\circ\mathcal{A}\circ(M^{\psi\varphi}\circ\varphi^{-1},\ \mathbb{1})$$

$$\partial_j\,\hat\psi_i\circ\hat\varphi^{-1} = \left(\underbrace{[\partial_k(\pi_{i-n})]}_{\delta_{k,i-n}}\circ\mathcal{A}\circ(M^{\psi\varphi}\circ\varphi^{-1},\ \mathbb{1})\right)\ \partial_j\left[\mathcal{A}\circ(M^{\psi\varphi}\circ\varphi^{-1},\ \mathbb{1})\right]_k$$

$$= \partial_j\,\mathcal{A}_{i-n}\circ(M^{\psi\varphi}\circ\varphi^{-1},\ \mathbb{1}) =$$

$$\underset{j\in \mathrm{I}}{=} A_{i-n}\circ\left(\partial_j\left(M^{\psi\varphi}\circ\varphi^{-1}\right),\ \mathbb{1}\right)$$

(plug in example to see)

$$= A_{i-n}\circ\left(d_j^{\varphi}(M^{\psi\varphi})\circ\varphi^{-1},\ \mathbb{1}\right)$$

$$\Rightarrow\ d_j^{\hat\varphi}(\hat\psi_i) = \left[\partial_j\left(\hat\psi_i\circ\hat\varphi^{-1}\right)\right]\circ\hat\varphi = A_{i-n}\circ\left(d_j^{\varphi}(M^{\psi\varphi})\circ\varphi^{-1},\ \mathbb{1}\right)\circ\hat\varphi$$

<u>Case 4:</u> $(i,j)\in \mathrm{II}^2$

plug in example into previous exp. with $j\in\mathrm{II}$,

Here we find $\partial_j\hat\psi_i\circ\hat\varphi^{-1} = \left(M^{\psi\varphi}\circ\varphi^{-1}\circ\pi_L\right)_{i-n,j-n}$

$$\Rightarrow\ d_j^{\hat\varphi}(\hat\psi_i) = \Big(M^{\psi\varphi}\circ\varphi^{-1}\circ\underbrace{\pi_L\circ\hat\varphi}_{\varphi\circ\pi}\Big)_{i-n,j-n} = \left(M^{\psi\varphi}\circ\pi\right)_{i-n,j-n}$$

All together in block form we write:

$$\boxed{M^{\hat\psi\hat\varphi} = \left[\begin{array}{c|c} M^{\psi\varphi}\circ\pi & 0 \\ \hline A_{i-n}\circ\left(d_j^{\varphi}(M^{\psi\varphi})\circ\varphi^{-1},\ \mathbb{1}\right)\circ\hat\varphi & M^{\psi\varphi}\circ\pi \end{array}\right]}$$

b) Let $\gamma:\mathbb{R}\to\mathcal{M}$ be a curve.

We know this induces tangent vectors as follows:

$\forall\ t\in\mathbb{R}$, if $f\in\mathcal{F}(\mathcal{M})$, $\partial_{t'}\big|_{t'=t}(f\circ\gamma)(t')\in\mathbb{R}$ is a tangent vector at $T_{\gamma(t)}\mathcal{M}$. We denote it by $\dot\gamma = \partial_{t'}(\cdot\circ\gamma)(t')$

$\forall\ t\in\mathbb{R}$, let $X(t)\in T_{\gamma(t)}\mathcal{M}$. So $\{X(t)\}_{t\in\mathbb{R}}$ is a family of tangent vectors along γ.

Define $\hat\gamma(t) := (\gamma(t), X(t))\in T\mathcal{M}$. $\hat\gamma$ is also a curve in a manifold — in $T\mathcal{M}$. This curve also induces tangent vectors, in $T_{\hat\gamma(t)}(T\mathcal{M})$ given by $\dot{\hat\gamma} = \partial_{t'}(\cdot\ \hat\gamma)(t')$

In the chart φ, γ has components $\{\varphi_i\circ\gamma\}_{i=1}^n$

$\dot\gamma$ has components $\{\dot\gamma(\varphi_i)\}_{i=1}^n = \{\partial_{t'}(\varphi_i\circ\gamma)(t')\}_{i=1}^n$

In the chart $\hat\varphi$, $\hat\gamma$ has components

$$\dot{\hat\gamma}(\hat\varphi_i) \equiv \partial_{t'}\left(\hat\varphi_i\circ\hat\gamma\right)(t')$$

if $i \in \mathrm{I}$, we get $\partial_{t'}(\varphi_i \circ \gamma)(t') \equiv \dot{\gamma}(\varphi_i)$

if $i \in \mathrm{II}$, $\partial_{t'}(X(\varphi_i))(t') \equiv \dot{X}(\varphi_i)$

$\Rightarrow \quad \dot{\hat{\gamma}}(\hat{\varphi}_i) = (\dot{\gamma}(\varphi_i), \dot{X}(\varphi_i))$

Def.: A parallel transport is a family of maps σ_Z (the parameter of the family is a point $Z \in TM$) s.t.

$$\sigma_Z : T_{\pi(Z)}M \longrightarrow T_Z(TM)$$

s.t. ① σ_Z is linear,

② $\pi_* \circ \sigma_Z = \mathbb{1}_{TM}$

where $\pi_* : T(TM) \to TM$ is the push-forward of the map $\pi : TM \to M$ given by $(p, v) \mapsto p$.

Hence if $Q \in T(TM)$ then $Q = ((p, v), \tilde{Q})$ with $p \in M$, $v \in T_pM$, $\tilde{Q} \in T_{(p,v)}(TM)$ and then $\pi_*(Q) \equiv (p, \tilde{Q}(\cdot \circ \pi))$

③ It is linear in Z (on the level of vector bundles) for fixed argument Y.

ii) **Cl.:** The 2^{nd} part of the expansion coeff. of $\sigma_Z(Y)$ w.r.t. the basis of the chart $\hat{\varphi}$ is linear in the exp. coeff. of the vec. part of Z and of Y. and called the Christoffel symbols. The 1^{st} is Y's exp. coeff.

Pf.: Let $Z \in TM$, $Y \in T_{\pi(Z)}M$ be given.

We know $(\pi_* \circ \sigma_Z)(Y) = Y$.

$\sigma_Z(Y) \in T_Z(TM)$, so we can expand it with $\{d\hat{\varphi}_i\}_{i=1}^{2n}$ at Z.

$$(\sigma_Z(Y))_i^{\hat{\varphi}} \equiv \{(\sigma_Z(Y))(\hat{\varphi}_i)\}_{i=1}^{2n} =: (\underbrace{S^L(Z,Y)}_{1^{st}\text{ part}}, \underbrace{S^R(Z,Y)}_{2^{nd}\text{ part}})$$

with $S^L(Z,Y) \in \mathbb{R}^n$, $S^R(Z,Y) \in \mathbb{R}^n$.

$\pi_*(\sigma_Z(Y)) = Y$

$\Rightarrow \quad Y_i^{\varphi} \equiv Y(\varphi_i) = (\pi_*(\sigma_Z(Y)))(\varphi_i) = (\sigma_Z(Y))(\varphi_i \circ \pi) \equiv (S^L(Z,Y))_i$

We now need to show $S^2(Z,Y)$ is linear in Z and Y.

$S^R(Z,Y) \equiv \left\{ (\sigma_Z(Y))_i^{\hat{\varphi}} \right\}_{i=n+1}^{2n}$

Since $\sigma_Z(Y)$ is linear in Y by def. & the coordinate functions $(\cdot)_i^{\hat{\varphi}}$ are linear, $S^R(Z,Y)$ is linear in Y.

Write $Z = (\pi(Z), z)$ with $z \in T_{\pi(Z)}M$.

We now want to show $S^R(Z,Y)$ is linear in $\{z_i^{\varphi}\}_{i=1}^{n}$.
But again since we assumed ③ and $(\cdot)_i^{\varphi}$ is linear, we find $S^R(Z,Y)$ is linear in z_i^{φ} as desired.

We define $\Gamma(p)$ $\forall\, p \in M$ via:

$$\boxed{-S^R(Z,Y)_i :=\ \Gamma(p)_{ijk}\, Y_j^{\varphi}\, z_k^{\varphi}} \quad \forall\, i \in \{1,\dots,n\}$$

(Linearity implies this def. makes sense)

c) We go back to the setting in b) where $\gamma: \mathbb{R} \to M$ was a curve, $\hat{\gamma}: \mathbb{R} \to TM$ is a curve "over" γ: We wrote

$$\hat{\gamma}(t) = (\underset{\in M}{\gamma(t)}, \underset{\in T_{\gamma(t)}M}{X(t)}) \in TM$$

__Def.:__ Given a parallel transport σ_η corresponding to a curve $\eta: \mathbb{R} \to TM$, we say that η is transported in a parallel fashion along the curve γ iff:

① $\pi \circ \eta = \gamma$

② $\dot{\eta} = \sigma_\eta(\dot{\gamma})$

iii) __Cl.:__ The above def. is equivalent to the usual definition, which says that a a family of vectors $Z(t) \in T_{\gamma(t)}M$ is parallel transported along γ, iff, given the parallel transport which is specified in a chart by the Christoffel symbols Γ, we have

$$\dot{Z}(\varphi_i) = -\Gamma_{ijk}(\gamma(t))\, \dot{\gamma}(\varphi_j)\, Z(\varphi_k)$$

Pf.: The first condition merely means that the curve η is "above" γ. It is equivalent to $Z(t) \in T_{\gamma(t)}M$ $\forall\, t \in \mathbb{R}$. We write the second condition in components:

$$\eta(t) = (\underset{\substack{\cap \\ M}}{\gamma(t)},\ \underset{\substack{\cap \\ T_{\gamma(t)}M}}{X(t)})$$

$$\dot\eta(t) = (\underset{\substack{\cap \\ T_{\gamma(t)}M}}{\dot\gamma(t)},\ \underset{\substack{\uparrow \\ \text{not in the} \\ \text{tangent space}}}{\dot X(t)}) \in T_{\eta(t)}(TM)$$

In a chart, evaluated @ t, but t suppressed

$$\dot\eta(\hat\varphi_i) = (\underset{\substack{\cap \\ \mathbb{R}^n}}{\dot\gamma(\varphi_i)},\ \underset{\substack{\cap \\ \mathbb{R}^n}}{\dot X(\varphi_i)}) \qquad \circledast$$

$$(\sigma_\eta(\dot\gamma))(\hat\varphi_i) \underset{\substack{\uparrow \\ \text{same notation} \\ \text{as before}}}{=} (\underset{\substack{\cap \\ \mathbb{R}^n}}{S^L(\eta,\dot\gamma)},\ \underset{\substack{\cap \\ \mathbb{R}^n}}{S^R(\eta,\dot\gamma)}) \in \mathbb{R}^{2n}$$

We saw $(S^L(\eta,\dot\gamma))_i = \dot\gamma(\varphi_i)$ which is compatible with $\circledast$. We've also defined Γ s.t.

$$-S^R(\eta,\dot\gamma)_i = \Gamma_{ijk}\,\dot\gamma(\varphi_j)\,X(\varphi_k)$$

So from $\circledast$ we get

$$\boxed{\dot X(\varphi_i) = -\Gamma_{ijk}\,\dot\gamma(\varphi_j)\,X(\varphi_k)}$$

as desired.

ii) Since σ is a tensor-valued map, we see that in this enlarged setting parallel transport <u>is</u> defined naturally with no reference to a chart. The Christoffel symbols however transform <u>not</u> like a tensor as they are only <u>part</u> of the components of a full tensor.

Let us see this explicitly:

$\sigma_Z(Y) \in T_Z(TM)$ for any $Y \in T_{\pi(Z)}M$, $Z \in TM$.

Thus we have the components of the "tangent vector" $\sigma_Z(Y)$

as $\quad (\sigma_2(Y))^{\hat\varphi}_i \equiv (\sigma_2(Y))(\hat\varphi_i)$

and we also know the transf. law:

$$(\overline{\sigma_2(Y)})^{\hat\psi}_i = M^{\hat\psi\hat\varphi}_{ij}\,(\sigma_2(Y))^{\hat\varphi}_j$$

We are interested in the Christoffel symbols, so $i\in\mathrm{II}$.

$$\begin{bmatrix}\overline{S^L}\\ \overline{S^R}\end{bmatrix} = \begin{bmatrix}\hat M_{11} & \hat M_{12}\\ \hat M_{21} & \hat M_{22}\end{bmatrix}\begin{bmatrix}S^L\\ S^R\end{bmatrix} = \begin{bmatrix}\hat M_{11}S^L + \hat M_{12}S^R\\ \hat M_{21}S^L + \hat M_{22}S^R\end{bmatrix}$$

Recall we found $\hat M_{22} = M^{\psi\varphi}\circ\pi$ and $\hat M_{21}$ was related to the derivatives of $M^{\psi\varphi}$ itself.

$$\overline{S^R}_i = \left(\mathcal{A}_i \circ (d_j(M)\circ\varphi^{-1}, \mathbb{1})\circ\hat\varphi\right) S^L_j + (M\circ\pi)_{ij}\, S^R_j$$

Recall $S^L_j = Y(\varphi_j)$ so the "complicated" first term becomes:

$$d_j(M_{ik})\, z_j\, Y_k$$

$$\Rightarrow \underbrace{\overline{S^R}_i}_{\underbrace{-\overline{\Gamma}_{imn}\, \overline{z}_m\, \overline{Y}_n}_{-\overline{\Gamma}_{imn} M_{mm'} M_{nn'} z_{m'} Y_{n'}}} = d_j(M_{ik})\, z_j\, Y_k - M_{ij}\,\Gamma_{jkl}\, z_k Y_l$$

$$\Rightarrow \left(-\overline{\Gamma}_{imn} M_{mm'} M_{nn'} - d_{m'}(M_{in'}) + M_{ij}\Gamma_{jm'n'}\right) z_{m'} Y_{l'} = 0$$

$$\Rightarrow \overline{\Gamma}_{im'n'} M_{m'm} M_{n'n} + d_m(M_{in}) = M_{ij}\Gamma_{jmn}$$

$$\Rightarrow \overline{\Gamma}_{imn} = M^{-1}_{mm'} M^{-1}_{nn'}\left(M_{ij}\Gamma_{jm'n'} - d_{m'}(M_{in'})\right)$$

$$= M_{ij}\Gamma_{jm'n'} M^{-1}_{mm'} M^{-1}_{nn'} + d_{m'}(M^{-1}_{nn'}) M_{in'} M^{-1}_{mm'}$$