Q1 Affine Connections

Let $\nabla : \Gamma(TM)^2 \to \Gamma(TM)$ be a given affine connection (so it satisfies the axioms on the bottom of pp. 18 in the lecture notes). (Recall $\Gamma(TM)$ is the $\mathcal{F}(M)$-module of sections on TM, or of vector fields on M).

Let $\delta : \Gamma(TM)^2 \to \Gamma(TM)$ be another given map (not necessarily an affine connection.

Define $B : \Gamma(TM)^2 \to \Gamma(TM)$ via $\boxed{B := \nabla - \delta}$.

Note This makes sense as $\Gamma(TM)$ is an $\mathcal{F}(M)$-module, so that its points may be added or subtracted.

Define a map $Q : \Gamma(T^*M) \times \Gamma(TM)^2 \to \mathcal{F}(M)$ via

$$(\omega, X, Y) \longmapsto \omega(B(X,Y))$$

Cl: δ is an affine connection $\boxed{\Longleftrightarrow}$ $Q \in \Gamma(T^*M \otimes T^*M \otimes T^*M)$, that is, Q is a tensor field of type $(1,2)$.

Pf: Recall $Q \in \Gamma(TM \otimes T^*M \otimes T^*M)$ iff it is an $\mathcal{F}(M)$-multilinear map in each of its slots.

For Q to be $\mathcal{F}(M)$-linear in its 1st slot we should have:

① $Q(f\omega, X, Y) \overset{?}{=} f\,Q(\omega, X, Y)$ $\quad \forall \; f \in \mathcal{F}(M)$.

But
$$\begin{aligned} Q(f\omega, X, Y) &\equiv (f\omega)(B(X,Y)) \\ &= f\,\omega(B(X,Y)) \\ &\equiv f\,Q(B(X,Y)) \end{aligned}$$

② $Q(\omega_1+\omega_2, X, Y) \overset{?}{=} Q(\omega_1, X, Y) + Q(\omega_2, X, Y)$

But
$$\begin{aligned} Q(\omega_1+\omega_2, X, Y) &\equiv (\omega_1+\omega_2)(B(X,Y)) \\ &\equiv \omega_1(B(X,Y)) + \omega_2(B(X,Y)) \\ &\equiv Q(\omega_1, X, Y) + Q(\omega_2, X, Y) \end{aligned}$$

So apparently Q is always $\mathcal{F}(M)$-linear in its first slot, regardless of what B is. ✓

Next note that Q is $\mathcal{F}(M)$-linear in its 2nd slot iff B is $\mathcal{F}(M)$-linear in its 1st slot (by $\mathcal{F}(M)$-linearity of ω).

Since ∇ is $\mathcal{F}(M)$-linear in its 1st slot and $\nabla = B + \delta$, B is $\mathcal{F}(M)$-linear in its 1st slot iff δ is $\mathcal{F}(M)$-lin. in its 1st slot, which is the (i) and $\frac{1}{2}$(ii) of the axioms for δ to be an affine connection.

[2]

Finally, Q is $\mathcal{F}(\mathcal{M})$-linear in its 2^{nd} slot iff B is $\mathcal{F}(\mathcal{M})$-lin. in its 2^{nd} slot. But note that

$$\nabla_X Y - \delta_X Y = B(X,Y)$$

$$\Rightarrow \begin{cases} f\nabla_X Y - f\delta_X Y = fB(X,Y) \\ \nabla_X fY - \delta_X fY = B(X,fY) \end{cases}$$

But we know that ∇ is an affine connection, so from its (iii)-axiom $\nabla_X fY = (X(f))Y + f\nabla_X Y$

$$\Rightarrow \begin{cases} f\nabla_X Y - f\delta_X Y = fB(X,Y) \\ X(f)Y + f\nabla_X Y - \delta_X fY = B(X,fY) \end{cases}$$

$$\Leftrightarrow \quad B(X,fY) - fB(X,Y) = X(f)Y + f\delta_X Y - \delta_X fY$$

$$\Leftrightarrow \quad \underbrace{L.H.S = 0}_{\substack{B \text{ is } \mathcal{F}(\mathcal{M})\text{-lin.} \\ \text{in } 2^{nd} \text{ slot}}} \text{ iff } \underbrace{R.H.S = 0}_{\substack{\delta \text{ obeys axiom} \\ (iii)}}$$

We find (separately for each axiom) that Q is $\mathcal{F}(\mathcal{M})$-linear in its 2^{nd} & 3^{rd} slots iff δ is also an affine connection.

An equivalent claim:

Cl.: The space of affine connections on $\mathcal{M}$ forms an affine space over the vector space of $\mathcal{F}(\mathcal{M})$-bilinear maps $\Gamma(T\mathcal{M})^2 \to \Gamma(T\mathcal{M})$.

Pf.: Recall $\mathcal{A}$ is an affine space over the v/sp. V iff $\exists$ gp. morphism $t: V \to \mathrm{Sym}(\mathcal{A})$ (where V is viewed as a gp. w.r.t. its additive structure and $\mathrm{Sym}(\mathcal{A})$ is the gp. of bijections $\mathcal{A}\circlearrowleft$) s.t. $\forall\, a \in \mathcal{A}$, $V \ni v \mapsto t(v)a \in \mathcal{A}$ is bijective.

Let $\mathcal{A}(\mathcal{M})$ be the sp. of affine connections on $\mathcal{M}$, and $V(\mathcal{M})$ be the v/sp. of $\mathcal{F}(\mathcal{M})$-bilinear maps $\Gamma(T\mathcal{M})^2 \to \Gamma(T\mathcal{M})$.

Define $t: V(\mathcal{M}) \to \mathrm{Aut}(\mathcal{A}(\mathcal{M}))$ via

$$B \mapsto (\cdot \mapsto \cdot - B)$$

Verify it's a gp. morphism, that it's well-defined essentially by the proof above and it is bijective again by the same proof.

Cl.: If $(\nabla, \delta) \in \mathcal{A}(\mathcal{M})^2$ then $((1-\alpha)\nabla + \alpha\delta) \in \mathcal{A}(\mathcal{M})$ $\forall\ \alpha \in (0,1)$.

Pf.: Let B be as before w.r.t. ∇ and δ.

Note $\nabla - [(1-\alpha)\nabla + \alpha\delta] = \alpha(\nabla - \delta) \equiv \alpha B$

Since $B \in \mathcal{V}(\mathcal{M}) \iff \alpha B \in \mathcal{V}(\mathcal{M})$ (with $\mathcal{V}(\mathcal{M})$ as above) $\forall\ \alpha \in \mathbb{R}$, we apply the 1^{st} claim to get the result.

Let $\varphi: \mathcal{U} \to \mathbb{R}^n$ be a chart. Recall we found an expression for the Christoffel symbols w.r.t. φ of the parallel transport induced by a $\nabla \in \mathcal{A}(\mathcal{M})$ to be:

$$\Gamma^{\varphi}_{\nabla\, ijk} = e^{\varphi}_i\left(\nabla_{d^{\varphi}_j} d^{\varphi}_k\right)$$

where $\{e^{\varphi}_i\}_i$ was the chart-basis for $\Gamma(T^*\mathcal{M}|_{\mathcal{U}})$, and $\{d^{\varphi}_i\}_i$ ———— $\Gamma(T\mathcal{M}|_{\mathcal{U}})$.

Cl.: The components of Q, $Q^{\varphi}{}_{ijk}$, satisfy:

$$Q^{\varphi}{}_{ijk} = \Gamma^{\varphi}_{\nabla\, ijk} - \Gamma^{\varphi}_{\delta\, ijk}$$

Pf.:

$$\Gamma^{\varphi}_{\nabla\, ijk} - \Gamma^{\varphi}_{\delta\, ijk} = e^{\varphi}_i\left(\nabla_{d^{\varphi}_j} d^{\varphi}_k\right) - e^{\varphi}_i\left(\delta_{d^{\varphi}_j} d^{\varphi}_k\right)$$

$$\overset{e^{\varphi}_i \text{ linear}}{=} e^{\varphi}_i\left(\nabla_{d^{\varphi}_j} d^{\varphi}_k - \delta_{d^{\varphi}_j} d^{\varphi}_k\right)$$

$$\overset{\text{def. of } B}{=} e^{\varphi}_i\left(B(d^{\varphi}_j, d^{\varphi}_k)\right)$$

$$\equiv Q(e^{\varphi}_i, d^{\varphi}_j, d^{\varphi}_k) \equiv Q^{\varphi}{}_{ijk}$$

We see again that the difference of two Christoffel symbols of two different connections behaves like a tensor, just like the difference of the symbols in two different charts.

Q2 Euclidean Metric in Polar Coord.

$\mathcal{M} = \mathbb{R}^2$ with the chart $\varphi = \mathbb{1}_{\mathcal{M}}$.

The metric g is a point in $\Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ which obeys certain axioms.

Thus we may specify it as the components of a $(0,2)$-tensor in a chart-basis $\{e_i^\varphi\}_i$ as:

$$\{g(d_i^\varphi, d_j^\varphi)\}_{i,j=1}^{2}$$

So, a 2×2 matrix.

The axioms are that g should be symmetric & non-deg., for that components matrix that means it should be symmetric and invertible.

Define $\{g(d_i^\varphi, d_j^\varphi)\}_{i,j} := \mathbb{1}_{2\times 2}$, which satisfies these requirements.

Recall from the previous HW the polar coord. chart

$$\psi : \mathbb{R}^2 \setminus \{0\} \longrightarrow (0,\infty)\times[0,2\pi)$$

$$x \longmapsto \begin{bmatrix} \|x\| \\ \operatorname{arctg}(x_2/x_1) \end{bmatrix}$$

To compute the metric in polar coord. we apply the usual coord. transf.:

$$\underbrace{g(d_i^\psi, d_j^\psi)}_{g^\psi_{ij}} = N^{\psi\varphi}_{ii'} N^{\psi\varphi}_{jj'} \underbrace{g(d_{i'}^\varphi, d_{j'}^\varphi)}_{=\delta_{i',j'}} = N^{\psi\varphi}_{ij'} N^{\psi\varphi}_{jj'}$$

$$= \left(N^{\psi\varphi}(N^{\psi\varphi})^T\right)_{ij} = \left|(N^{\psi\varphi})^T\right|^2_{ij} \qquad (|A|^2 = A^T A)$$

We found then

$$N^{\psi\varphi}(\psi^{-1}(r,\varphi)) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -r\sin(\varphi) & r\cos(\varphi) \end{bmatrix}$$

$$\Rightarrow g^\psi(\psi^{-1}(r,\varphi)) = \left[\quad\right]\left[\quad\right]^T = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

The Christoffel symbols of the g-Levi-Civita connection are given by the formula

$$\Gamma^{\varphi}{}_{ijk} = \frac{1}{2}(g^{\varphi})^{-1}{}_{il}(g^{\varphi}{}_{jl,k} + g^{\varphi}{}_{kl,j} - g^{\varphi}{}_{jk,l})$$

Note $g^{\varphi}{}_{,r} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}$ $\quad g^{\varphi}{}_{,\varphi} = 0$

$(g^{\varphi})^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$ $\quad (g^{\varphi})^{-1} g^{\varphi}{}_{,\varphi} = 0$

$(g^{\varphi})^{-1} g^{\varphi}{}_{,r} = \begin{bmatrix} 0 & 0 \\ 0 & 2r^{-1} \end{bmatrix}$

$$\Gamma^{\varphi}{}_{ijk} = \frac{1}{2}(g^{\varphi})^{-1}{}_{ir}\,(g^{\varphi}{}_{jr,k}$$

 Q3

Torsion & the Hessian

$\nabla \in \mathcal{A}(\mathcal{M})$ affine connection

Cl.: $df = \nabla_{\cdot} f \qquad \forall\ f \in \mathcal{F}(\mathcal{M})$

Pf.: $f : \mathcal{M} \to \mathbb{R} \Rightarrow df : T_p\mathcal{M} \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$

$$X \mapsto X(\cdot\circ f) \equiv X(f)$$

O.T.O.H., $\nabla_{\cdot} f$ is a 1-form:

$$(\nabla_{\cdot} f)(X) \equiv \nabla_X f = X(f)$$

Def.: $\forall\ f \in \mathcal{F}(\mathcal{M})$, the Hessian H of f is

$$H(f) := \nabla(\nabla f) \qquad \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M}).$$

Recall the def. of torsion: (2.8)

Def.: Torsion T corresp. to $\nabla \in \mathcal{A}(\mathcal{M})$ is a map

$$\Gamma(T\mathcal{M})^2 \longrightarrow \Gamma(T\mathcal{M}) \quad \text{defined by}$$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

Cl.: $H(f)$ is symmetric $\Leftrightarrow T = 0$

Pf.: $(H(f))(X,Y) \equiv (\nabla(df))(X,Y)$

$$\underset{\substack{\uparrow \\ \text{slot of } \nabla_{\cdot} \\ \text{always takes} \\ \text{last argument}}}{=} (\nabla_Y df)(X) \underset{\substack{\uparrow \\ (2.8)}}{=} Y(X(f)) - (df)(\nabla_Y X)$$

$$= (YX - \nabla_Y X) f$$

$$\Rightarrow (H(f))(X,Y) - (H(f))(Y,X) = (YX - \nabla_Y X - XY + \nabla_X Y) f$$

$$= (\nabla_X Y - \nabla_Y X - [X,Y]) f$$

$$\equiv (T(X,Y))(f)$$

Q4

Geodesics in the Hyperbolic Plane

$\mathcal{M} := \{ x \in \mathbb{R}^2 \mid \pi_2(x) > 0 \}$

Chart $\varphi := \mathbb{1}_{\mathcal{M}}$ — $\pi_n : \mathbb{R}^m \to \mathbb{R}$ proj. onto n^{th} coord.

Metric $g^{\varphi}(x) = \pi_2(x)^{-2}\, \mathbb{1}_{2\times 2} \equiv (x_2)^{-2}\, \mathbb{1}_{2\times 2}$

Recall that a curve $\gamma : \mathbb{R} \to \mathcal{M}$ is called a g-geodesic iff $\boxed{\nabla \dot{\gamma} = 0}$ where ∇ is the g-Levi-Civita connection.

That implies in a chart $\nabla \dot{\gamma} = 0$ becomes ((3.10)):

$$\boxed{\ddot{\gamma}^{\varphi}_i + \Gamma^{\varphi}_{ijk}\, \dot{\gamma}^{\varphi}_j\, \dot{\gamma}^{\varphi}_k = 0}$$

Γ evaluated on γ!

We start by calculating Γ^{φ}:

$$\Gamma^{\varphi}{}_{ijk} = \frac{1}{2}((g^{\varphi})^{-1})_{il} \left[(g^{\varphi})_{jl,k} + (g^{\varphi})_{kl,j} - (g^{\varphi})_{jk,l} \right]$$

$$g^{\varphi}(x) = (x_2)^{-2}\, \mathbb{1}_{2\times 2}, \quad (g^{\varphi})^{-1}(x) = (x_2)^2\, \mathbb{1}_{2\times 2}$$

$$\Gamma^{\varphi}{}_{1jk}(x) = \frac{1}{2}(x_2)^2 \Big[\underbrace{\underbrace{(g^{\varphi})_{j1,k}}_{[(x_2)^{-2}\delta_{j,1}]_{,k}}}_{-2(x_2)^{-3}\delta_{j,1}\delta_{k,2}} + \underbrace{(g^{\varphi})_{k1,j}}_{-2(x_2)^{-3}\delta_{k,1}\delta_{j,2}} - \underbrace{(g^{\varphi})_{jk,1}}_{0} \Big]$$

$$= -\frac{1}{x_2}(\delta_{j,1}\delta_{k,2} + \delta_{j,2}\delta_{k,1})$$

$$\boxed{\Gamma^{\varphi}_{1\cdot\cdot}(x) = -\frac{1}{x_2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\Gamma^{\varphi}{}_{2jk}(x) = \frac{1}{2}(x_2)^2 \Big[\underbrace{\underbrace{(g^{\varphi})_{j2,k}}_{[(x_2)^{-2}\delta_{j,2}]_{,k}}}_{-2(x_2)^{-3}\delta_{j,2}\delta_{k,2}} + (g^{\varphi})_{k2,j} - \underbrace{(g^{\varphi})_{jk,2}}_{-2(x_2)^{-3}[\delta_{j,1}\delta_{k,1} + \delta_{j,2}\delta_{k,2}]} \Big]$$

$$= \frac{1}{x_2}\delta_{j,1}\delta_{k,1} - \frac{1}{x_2}\delta_{j,2}\delta_{k,2} \Rightarrow \boxed{\Gamma^{\varphi}_{2\cdot\cdot}(x) = \frac{1}{x_2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

We go on to calculate the geodesic eq-n:

$$\ddot{\gamma}^{\varphi}_1 + \begin{bmatrix} \dot{\gamma}^{\varphi}_1 & \dot{\gamma}^{\varphi}_2 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\gamma^{\varphi}_2} \\ -\frac{1}{\gamma^{\varphi}_2} & 0 \end{bmatrix} \begin{bmatrix} \dot{\gamma}^{\varphi}_1 \\ \dot{\gamma}^{\varphi}_2 \end{bmatrix} = 0$$

$$\boxed{\gamma^{\varphi}_2 \ddot{\gamma}^{\varphi}_1 = 2 \dot{\gamma}^{\varphi}_1 \dot{\gamma}^{\varphi}_2} \quad (1)$$

(We omit the φ superscript...)

$$\ddot{\gamma}_2 + \begin{bmatrix} \dot{\gamma}_1 & \dot{\gamma}_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma_2} & 0 \\ 0 & -\frac{1}{\gamma_2} \end{bmatrix} \begin{bmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{bmatrix} = 0$$

$$\boxed{\gamma_2 \ddot{\gamma}_2 = (\dot{\gamma}_2)^2 - (\dot{\gamma}_1)^2} \quad (2)$$

Cl.: If $\dot{\gamma}_1 \neq 0$ then $(1) \wedge (2) \Rightarrow \gamma_1^2 + \gamma_2^2 - a\gamma_1 = b$ $\exists (a,b) \in \mathbb{R}^2$.

Pf.: $\dot{\gamma}_1 \neq 0 \Rightarrow (1)$ implies $\dfrac{\ddot{\gamma}_1}{\dot{\gamma}_1} = 2\dfrac{\dot{\gamma}_2}{\gamma_2}$

Note: $\dot{\log}(f) = \frac{1}{f}\dot{f} \Rightarrow \dot{\log}(\dot{\gamma}_1) = 2\,\dot{\log}(\gamma_2)$

$$\Rightarrow \log(\dot{\gamma}_1) = 2\log(\gamma_2) + q \quad \exists\, q \in \mathbb{R}$$

$$\Rightarrow \dot{\gamma}_1 = \exp(\log(\gamma_2^2) + q) = \gamma_2^2 \exp(q)$$

$$\boxed{\dot{\gamma}_1 = D\,\gamma_2^2} \qquad D := \exp(q)$$

Let $f : \mathbb{R} \to \mathbb{R}_{>0}$ be given by $\gamma_1(t) \mapsto \gamma_2(t)$
(We know we can find one such map for all t by reparametrizing γ_2 to be w.r.t. γ_1 instead of t)

Cl.: (2) implies $ff'' + (1+f'^2) = 0$

Pf.:

$$f' = \frac{\partial \gamma_2}{\partial \gamma_1} = \frac{\dot{\gamma}_2}{\dot{\gamma}_1}$$

$$f'' = \left(\frac{\dot{\gamma}_2}{\dot{\gamma}_1}\right)' = \frac{\ddot{\gamma}_2}{(\dot{\gamma}_1)^2} - \frac{\dot{\gamma}_2}{(\dot{\gamma}_1)^2}(\dot{\gamma}_1)'$$

$$(\dot{\gamma}_1)' = (Df^2)' = D\,2ff' = 2D\gamma_2 \frac{\dot{\gamma}_2}{\dot{\gamma}_1} = \frac{2\dot{\gamma}_2}{\gamma_2}$$

$$\Rightarrow ff'' + 1 + f'^2 = \frac{\gamma_2 \ddot{\gamma}_2}{\dot{\gamma}_1^2} - \frac{2\dot{\gamma}_2^2}{\dot{\gamma}_1^2} + 1 + \frac{\dot{\gamma}_2^2}{\dot{\gamma}_1^2} =$$

$$= \dot{\gamma}_1^{-2}(\gamma_2\ddot{\gamma}_2 - \dot{\gamma}_2^2 + \dot{\gamma}_1^2) \underset{\text{by } (2)}{=} 0$$

However $ff'' + 1 + f'^2 = 0$ may be solved via:

$$f(x) = \pm\sqrt{a^2 - (x-x_0)^2} \quad \text{(the $-$ solution is irrelevant)}$$

Indeed, $f'(x) = \frac{1}{2} f(x)^{-1}(-2(x-x_0)) = -\frac{x-x_0}{f(x)}$

$$f''(x) = -\frac{1}{f(x)} + \frac{x-x_0}{f(x)^2} f'(x) = -\frac{1}{f(x)} - \frac{(x-x_0)^2}{f(x)^3}$$

$$(ff'' + 1 + f'^2)(x) = \cancel{-1} - \cancel{\frac{(x-x_0)^2}{f(x)^2}} + \cancel{1} + \cancel{\frac{(x-x_0)^2}{f(x)^2}} = 0 \checkmark$$

We find the desired result.

This eq-n describes a semi-circle whose center is on the horizontal axis.

Note that if $\dot{\gamma}_1 = 0$, we find γ describes a vertical line.

Note that except for $\frac{\dot{\gamma}_1}{\gamma_2^2}$ being a constant there is another constant of motion:

Cl.: $\frac{\|\dot{\gamma}\|^2}{\gamma_2^2}$ is a constant of motion.

Pf.: The geodesic γ is the extremum of the Lagrangian

$$L(\gamma_1, \gamma_2, \dot{\gamma}_1, \dot{\gamma}_2) = \frac{1}{2}\frac{\|\dot{\gamma}\|^2}{\gamma_2^2}$$

Since L doesn't depend explicitly on time, its Legendre transform, the "energy", is const:

$$\boxed{\dot{\gamma}_2 \frac{\partial L}{\partial \dot{\gamma}_2} + \dot{\gamma}_1 \frac{\partial L}{\partial \dot{\gamma}_1} - L = L}$$

Later on you'll see another way to find conserved quantities via "Killing fields"!