Q1 (Alternative way to the one suggested by the hints)

In **Q2** we find that for a grav. field φ,

$$\Gamma_{i00} = \partial_i \varphi \qquad \forall \quad i \in \{1,2,3\}$$

Work in 2D spacetime. $\Rightarrow \quad \Gamma_{100} = \varphi'$

Assume metric doesn't dep. on time:

$$\Gamma_{100} = \tfrac{1}{2}(g^{-1})_{1\ell}(\underbrace{g_{0\ell,0}}_{=0} + \underbrace{g_{0\ell,0}}_{=0} - \underbrace{g_{00,\ell}}_{=g_{00,1}\delta_{\ell,1}})$$

$$= -\tfrac{1}{2}(g^{-1})_{11}\, g_{00,1}$$

Assume g is almost the Mikowski metric: $g = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} + h$

where h has "small" components.

$\Rightarrow \quad g^{-1} \approx \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + O(h) \quad , \quad g_{00,1} = h_{00,1} = h_{00}'$

$\Rightarrow \quad \boxed{\Gamma_{100} = +\tfrac{1}{2} h_{00,1}} \quad \Rightarrow \quad \tfrac{1}{2} h_{00}' = \varphi'$

$\Rightarrow \quad \boxed{g_{00} = 1 + 2\varphi}$ (for this choice of const)

If we pick a homogeneous grav. field, $\varphi(x) = \gamma x_1$ where γ is the gravitational const. (9.8) with γ instead of g to avoid conflict of notation with the metric g.

We know (eq-n (4.4) in the script) that

$$(\Delta\tau)^2 = g_{00}(x)(\Delta t)^2 \quad \Rightarrow \quad \boxed{\Delta\tau = \sqrt{1 + 2\gamma x_1}\,\Delta t}$$

where $\Delta\tau$ is the proper elapsed time of the clock, x_1 is its position, and Δt is the elapsed time in the chart.

We have one clock at height ℓ (q_2) and another clock at height zero (q_1) $\Rightarrow \quad \Delta\tau_1 = \Delta t \quad , \quad \Delta\tau_2 = \sqrt{1 + 2\gamma\ell}\,\Delta t$

$$\Rightarrow \quad \Delta\tau_1 = (1 + 2\gamma\ell)^{-1/2}\Delta\tau_2 \underset{\gamma\ell \ll 1}{\approx} (1 - \gamma\ell)\Delta\tau_2$$

Q2 Newton's Eq-n as a Geodesic Eq-n

In classical mechanics, when a particle is placed in a gravitational potential $\varphi : \mathbb{R}^3 \to \mathbb{R}$ its trajectory $\gamma : \mathbb{R} \to \mathbb{R}$ obeys the equation $\boxed{\ddot{\gamma} = -(\nabla\varphi)\circ\gamma}$. (*)

In 4D spacetime, a curve is given by a map $\tilde{\eta} : \mathbb{R} \to \mathcal{M}$ where the domain $\mathbb{R}$ is the parameter of the curve $\tilde{\eta}$ (not necessarily time) and $\mathcal{M}$ is the spacetime manifold, which by GR is a 4D pseudo-Riemannian manifold.

Hence we should have a chart $\psi : \mathcal{U} \to \mathbb{R}^4 \quad \exists\ \mathcal{U} \in \mathrm{Open}(\mathcal{M})$.

Then $\psi \circ \tilde{\eta}\big|_{\tilde{\eta}^{-1}(\mathcal{U})} : \tilde{\eta}^{-1}(\mathcal{U}) \to \mathbb{R}^4$ is a curve in $\mathbb{R}^4$ parametrized by an open subset of $\mathbb{R}$ (since $\tilde{\eta}$ is cont.).

$V := \tilde{\eta}^{-1}(\mathcal{U}) \in \mathrm{Open}(\mathbb{R})$

$\eta := \psi \circ \tilde{\eta}\big|_V$

η is hence a curve in $\mathbb{R}^4$.

If we want to "connect" η with γ then we could "declare" that $\eta_0 := \mathbb{1}$ (viewing the 0^{th} component of $\mathbb{R}^4$ as that which corresponds to time) and $(\eta_i)_{i=1,2,3} := \gamma$.

Then we can figure out which EoM η satisfies!

We let now the dot be the derivative w.r.t. the param. of the curve η, $\lambda \in V$: $\quad \dot{} \equiv \partial_\lambda$

$$\eta_0(\lambda) \equiv \lambda \Rightarrow \dot{\eta}_0(\lambda) = 1 \Rightarrow \ddot{\eta}_0(\lambda) = 0$$

from (*) we get:

$$(\partial_t{}^2\gamma)_i(t) = -((\nabla\varphi)\circ\gamma)(t) = -(\partial_i\varphi)(\gamma(t))$$

$$\Rightarrow \overbrace{(\partial_\lambda{}^2\eta)_i(\lambda)}^{\ddot{\eta}_i \equiv} + (\partial_i\varphi)(\eta(\lambda)) = 0$$

But $\dot{\eta}_0(\lambda) = 1$, so we may rewrite this as:

$$\boxed{\ddot{\eta}_i + [(\partial_i\varphi)\circ\eta]\,\dot{\eta}_0\,\dot{\eta}_0 = 0}$$

We then recall what the geodesic eq-n for a curve $\eta : V \to \mathbb{R}^4$ would look like given Christoffel symbols:

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$$\ddot{\eta}_i + \Gamma_{ijk} \dot{\eta}_j \dot{\eta}_k = 0$$

For the two equations to coincide, we may define

$$\boxed{\Gamma_{0jk} := 0} \quad \text{for} \quad \ddot{\eta}_0 = 0.$$

$$\boxed{\Gamma_{i00} := \partial_i \varphi} \quad \text{for the other three eq-ns,} \quad i \in \{1,2,3\}.$$

All other components should be zero.

Such a def. for Γ is clearly symmetric, giving (i). ✓

The corresponding Riemann curvature tensor, given by eq-n (2.15) in the script:

$$R_{ijkl} \overset{(2.15)}{=} \Gamma_{ilj,k} - \Gamma_{ikj,l} + \Gamma_{slj}\Gamma_{iks} - \Gamma_{skj}\Gamma_{ils} \quad (i,j,k,l) \in \{0,1,2,3\}^4$$

$$\Gamma_{slj}\Gamma_{iks} - \Gamma_{skj}\Gamma_{ils} = \underbrace{\Gamma_{0lj}}_{=0}\Gamma_{ik0} + \sum_{s=1}^{3} \Gamma_{slj}\underbrace{\Gamma_{iks}}_{=0} - \underbrace{\Gamma_{0kj}}_{=0}\Gamma_{il0} - \sum_{s=1}^{3}\Gamma_{skj}\Gamma_{ils}$$

$$= -\sum_{s=1}^{3}\Gamma_{skj}\Gamma_{ils} = 0 \quad \text{if } k \in \{1,2,3\}.$$

Also $\Gamma_{ikj,l} = 0$ if $k \in \{1,2,3\}$.

$$\Longrightarrow \quad R_{ijkl} = \Gamma_{ilj,k} \quad \text{if } k \in \{1,2,3\}.$$

$$\Longrightarrow \quad R_{i0k0} = \Gamma_{i00,k} = \partial_k \partial_i \varphi \quad \text{if } k,i \in \{1,2,3\}.$$

$$R_{ijk0} = \Gamma_{i0j,k} = 0 \quad \text{if } i,j,k \in \{1,2,3\}.$$

If Γ were to be associated with a Levi-Civita connection, we know it should obey the symmetries on pp. 29, namely:

$$\begin{cases} g_{ii'} R_{i'jkl} = -g_{jj'} R_{j'ikl} \\ g_{ii'} R_{i'jkl} = g_{kk'} R_{k'lij} \end{cases}$$

$$g_{ii'} R_{i'0k0} = g_{ii'} \partial_k \partial_{i'} \varphi \overset{?}{=} -g_{0j'} \underbrace{R_{j'ik0}}_{=0}$$

$\Longrightarrow$ Not true unless $\boxed{\partial_k \partial_i \varphi = 0} \Longleftrightarrow$ Const. grav. field.

Q3. On the Levi-Civita Connection

Let $\mathcal{M}$ be a manifold.

Def.: A subset $\mathcal{N} \subseteq \mathcal{M}$ is called an embedded submanifold of $\mathcal{M}$ iff $\forall\, p \in \mathcal{N}\ \exists$ chart $\psi : \mathcal{U} \to \mathbb{R}^m$ ($\mathcal{U} \in \mathrm{Open}(\mathcal{M})$, $p \in \mathcal{U}$) s.t. $\psi(\mathcal{U} \cap \mathcal{N}) = \psi(\mathcal{U}) \cap \mathbb{W}$ where $\mathbb{W} \subseteq \mathbb{R}^m$ is a subspace of dim. $n \equiv \dim(\mathcal{N})$, $0 \leq n \leq m$. The atlas for $\mathcal{N}$ is the pairs $\mathcal{U} \cap \mathcal{N}$, $\psi|_{\mathcal{U} \cap \mathcal{N}}$.

$\imath : \mathcal{N} \hookrightarrow \mathcal{M}$, the inclusion map, induces a tangent map:
$(d\imath)_p : T_p\mathcal{N} \hookrightarrow T_p\mathcal{M}$, which is injective as well: (We also use $\imath_* \equiv d\imath$)

$$(d\imath)_p(v) = (d\imath)_p(u)$$
$$\Updownarrow$$
$$v(\cdot \circ \imath) = u(\cdot \circ \imath)$$
$$\Updownarrow$$
$$v(f \circ \imath) = u(f \circ \imath) \quad \forall\ f \in \mathcal{F}(\mathcal{M})$$

But since $\forall\ g \in \mathcal{F}(\mathcal{N})$, $\exists$ extension $\tilde{g} \in \mathcal{F}(\mathcal{M})$: $\tilde{g} \circ \imath = g$, $v = u$.

$\Longrightarrow$ We think of $T_p\mathcal{N}$ as a linear subsp. of $T_p\mathcal{M}$.

If $g \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ is a metric on $\mathcal{M}$, a $\not\!\!\!\!\!\!\rightharpoonup$ metric $h \in \Gamma(T^*\mathcal{N} \otimes T^*\mathcal{N})$ is induced via the eq-n:

$$h(X, Y) := g((d\imath)(X), (d\imath)(Y))$$

$$\forall\ (X, Y) \in \Gamma(T\mathcal{N})^2.$$

$\forall\ p \in \mathcal{N}$, define $P_p : T_p\mathcal{M} \to T_p\mathcal{N}$ as the orthogonal proj. assoc. to g_p: g_p is an inner-prod. on $T_p\mathcal{M}$, so it defines what it means for two vectors to be orthogonal:

$$\left[\, u \perp_{g_p} v \iff g_p(u, v) = 0 \,\right] \quad \forall\ (u, v) \in (T_p\mathcal{M})^2$$

Then the orthogonal projection P_p satisfies:

$$P_p u = u \quad \forall\ u \in T_p\mathcal{N} \quad \text{bcs. } P_p \text{ is a proj.}$$

$$g_p(v, (d\imath)u) = g_p(v, P_p(d\imath)u) = g_p(P_p v, (d\imath)u)$$

bcs. it is an orthogonal proj.

$$\forall\ u \in T_p\mathcal{N},\ v \in T_p\mathcal{M}.$$

6 Let ∇ be the g-Levi-Civita connection.
— δ — h — .

<u>Cl.</u>: $(\delta_X Y)_p = \tilde{P}_p(\nabla_{\iota_* X} \iota_* Y) \quad \forall \quad (X,Y) \in \Gamma(TN)^2, \ p \in N.$

with $\tilde{P}$ being the proj. P w/ its codomain TN.

<u>Pf.</u>: Let $(X,Y) \in \Gamma(TN)^2$, $p \in N$.

<u>Cl.</u>: $\tilde{P} \circ \nabla_{\iota_* \cdot} \iota_* \cdot$ is an affine connection on N.

<u>Pf.</u>: (i) W.T.S. $\tilde{P} \circ \nabla_{\iota_* X} \iota_* Y$ is $\mathcal{F}(N)$-lin. in X:

$$\tilde{P} \circ \nabla_{\iota_*(X_1+X_2)} \iota_* Y \overset{\iota_* \text{ linear}}{=} \tilde{P} \circ \nabla_{\iota_* X_1 + \iota_* X_2} \iota_* Y$$

$$\overset{\nabla \text{ is an affine conn.}}{=} \tilde{P} \circ (\nabla_{\iota_* X_1} \iota_* Y + \nabla_{\iota_* X_2} \iota_* Y)$$

$$\overset{\tilde{P} \text{ is linear}}{=} \tilde{P} \circ \nabla_{\iota_* X_1} \iota_* Y + \tilde{P} \circ \nabla_{\iota_* X_2} \iota_* Y \quad \checkmark$$

$$(\tilde{P} \circ \nabla_{\iota_* f X} \iota_* Y)_p \overset{\iota_* f X = f \iota_* X}{=} (\tilde{P} \circ \nabla_{f \iota_* X} \iota_* Y)_p \qquad f \in \mathcal{F}(N),\ p \in N$$

$$= (\tilde{P} \circ f \, \nabla_{\iota_* X} \iota_* Y)_p$$

$$= \tilde{P}_p (f(p) \nabla_{\iota_* X} \iota_* Y)_p$$

$$\overset{\tilde{P}_p \text{ is linear}}{=} f(p) \tilde{P}_p (\nabla_{\iota_* X} \iota_* Y)_p \quad \checkmark$$

(ii) W.T.S. $\tilde{P} \circ \nabla_{\iota_* X} \iota_* Y$ is $\mathbb{R}$-lin. in Y:

$Y_1 + Y_2$ follows by lin. of ι_*, ∇ being affine connection and P being linear.

(iii) W.T.S. $\tilde{P} \circ \nabla_{\iota_* X} \iota_* Y$ obeys Leibniz in Y:

$$(\tilde{P} \circ \nabla_{\iota_* X} \iota_* f Y)_p = \tilde{P}_p \circ (\nabla_{\iota_* X} \tilde{f} \iota_* Y)_p =$$

$\tilde{f}$ extends f to M (trivially)

$$= \tilde{P}_p \circ (f(p) \nabla_{\iota_* X} \iota_* Y + \iota_* Y (\iota_* X)(\tilde{f}))_p$$

$$= f(p)\,\tilde{P}_p(\nabla_{\iota_* X}\iota_* Y)_p + (\tilde{P}_p(\iota_* Y)_p)(\iota_* X)(f)$$

$\tilde{P}_p \circ (\iota_*)_p = \mathbb{1}$

$$= f(p)\,\tilde{P}_p(\nabla_{\iota_* X}\iota_* Y)_p + (Y)_p(\iota_* X)(f)$$

$\Longrightarrow$ $\tilde{P}\circ\nabla_{\iota_*\cdot}\,\iota_*\cdot$ is an affine conn. on N. ✓

Cl.: The torsion of is zero.

Pf.: The torsion is given by:

$$\tilde{P}\nabla_{\iota_* X}\iota_* Y - \tilde{P}\nabla_{\iota_* Y}\iota_* X - [X,Y] =$$

$$\stackrel{\tilde{P}\circ\iota_* = \mathbb{1}}{=} \tilde{P}\left(\nabla_{\iota_* X}\iota_* Y - \nabla_{\iota_* Y}\iota_* X - \iota_*[X,Y]\right)$$

Cl.: $\iota_*[X,Y] = [\iota_* X, \iota_* Y]$

Pf.: $(\iota_*[X,Y])(f) \equiv ([X,Y])(\iota\circ f)$

$= X(Y(\iota\circ f)) - Y(X(\iota\circ f))$

$([\iota_* X, \iota_* Y])(f) \equiv (\iota_* X)(Y(\iota\circ f)) - \dots$

$\equiv X(Y_\iota(\iota\circ f)) - \dots$

$\iota(p) \equiv p \ \forall p\in N \quad \equiv X(Y(\iota\circ f))$

$= \tilde{P}\,0$ as ∇ has no torsion.

Cl.: $(\tilde{P}\circ\nabla_{\iota_*})(h) = 0$

Pf.: We use the inductive formula:

$$(\nabla_Z T)(X_1,\dots,X_k,\mu_1,\dots,\mu_l) = \overbrace{\nabla_Z(T(X_1,\dots,X_k,\mu_1,\dots,\mu_l))}^{Z(T(X_1,\dots,X_k,\mu_1,\dots,\mu_l))}$$
$$- T(\nabla_Z X_1, X_2,\dots,X_k,\mu_1,\dots,\mu_l)$$
$$\dots$$
$$- T(X_1,\dots,X_k,\mu_1,\dots,\nabla_Z \mu_l)$$

to get:

$$(\tilde{P}\nabla_{\iota_* Z} h)(X,Y) = Z(h(X,Y)) - h(\tilde{P}\nabla_{\iota_* Z}\iota_* X, Y) - h(X, \tilde{P}\nabla_{\iota_* Z}\iota_* Y)$$

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$$= \underbrace{\iota^* Z}_{} (g(\iota_* X, \iota_* Y)) - g(\underbrace{\iota_* \tilde{P}}_{P} \nabla_{\iota_* Z} \iota_* X, \iota_* Y) - g(\iota_* X, \underbrace{\iota_* \tilde{P}}_{P} \nabla_{\iota_* Z} \iota_* Y)$$

bcs. $\iota(p) = p \ \forall p \in N$

P orthogonal proj.

$$= \iota_* Z (g(\iota_* X, \iota_* Y)) - g(\nabla_{\iota_* Z} \iota_* X, \underbrace{P \iota_*}_{= \iota_*} Y) - g(\underbrace{P \iota_*}_{= \iota_*} X, \nabla_{\iota_* Z} \iota_* Y)$$

$$= (\nabla_{\iota_* Z}\, g)(\iota_* X, \iota_* Y)$$

$$= 0 \quad \text{in particular.}$$

Hence by uniqueness of the Levi-Civita connection we obtain the result.