(Alternative way to use one suggested by the hint)

In (15) we find that for a gauge field \( A \),

\[
\Pi_{10} = \partial_t A^t \quad \forall \, \delta x^1, \delta x^2
\]

Work in 2D spacetime. \( \Rightarrow \Pi_{10} = A^t \)

Assume metric doesn't depend on time:

\[
\Pi_{10} = \frac{1}{2} \partial_t (g^{11} + g^{12} - g^{01} \delta e_1)
\]

\[
= -\frac{1}{2} (g^{11} + g^{12}) \delta e_{10}
\]

Assume \( g \) is almost the Minkowski metric:

\[
\Pi_{10} = \frac{1}{2} \Pi_{10,1} \Rightarrow \frac{1}{2} \Pi_{10,1} = A^t \Rightarrow \Pi_{10} = 1 + 2A^t
\]

If we pick a homogeneous gauge field, \( A^t = \lambda \partial t \), where \( \lambda \) is the gravitational constant (9.8) with \( \gamma \) instead of \( g \) to avoid conflict of notation with the metric \( g \).

We have (eq. 13 in the script) that

\[
(\Delta \tau)^2 = g_{00}(x) (\Delta t)^2 \Rightarrow \Delta \tau = \sqrt{1 + 2\lambda x^1} \Delta t
\]

Where \( \Delta \tau \) is the proper time of the clock, \( x^1 \) is its position, and \( \Delta t \) is the elapsed time in the chart.

We have one clock at height \( (9.2) \) and another clock at height zero \((0.0)\): \( \Delta \tau_1 = \Delta t \), \( \Delta \tau_2 = \sqrt{1 + 2\lambda x^1} \Delta t \)

\[
\Rightarrow \Delta \tau_1 = (1 + 2\lambda \delta x^1) \Delta \tau_2 \quad \forall \, \delta x^1 \ll 1
\]
In classical mechanics, when a particle is placed in a gravitational potential \( V: \mathbb{R}^3 \to \mathbb{R} \), its trajectory \( \gamma: \mathbb{R} \to \mathbb{R}^3 \) obeys the equation \[ \ddot{\gamma} = -(V' \circ \gamma) \hat{\gamma}. \]

In 4D spacetime, a curve is given by a map \( \tilde{\gamma}: \mathbb{R} \to \mathbb{R}^4 \) where the domain \( \mathbb{R} \) is the parameter of the curve \( \tilde{\gamma} \) (not necessarily time) and \( \mathbb{R}^4 \) is the spacetime manifold, which by GR is a 4D pseudo-Riemannian manifold.

Hence we should have a chart \( \psi: U \to \mathbb{R}^3 \equiv U \cap \text{Open}(\mathbb{R}^3) \).

Then \( \psi \circ \tilde{\gamma} |_{\psi(U)} : \tilde{\gamma}^{-1}(\psi(U)) \to \mathbb{R}^3 \) is a curve in \( \mathbb{R}^3 \) parametrized by an open subset of \( \mathbb{R} \) (since \( \tilde{\gamma} \) is cont).

Let \( V := \tilde{\gamma}^{-1}(\psi(U)) \cap \text{Open}(\mathbb{R}^3) \).

\( \eta := \psi \circ \tilde{\gamma} |_{V} \)

\( \eta \) is hence a curve in \( \mathbb{R}^3 \).

If we want to "connect" \( \eta \) with \( \gamma \) then we could "declare" that \( \gamma_0 := \gamma \) (viewing the 0th component of \( \mathbb{R}^3 \) as that which corresponds to time) and \( (\eta_0)_{\mathbb{R}^4} |_{V} := \gamma_0 \).

Then we can figure out which EoM \( \eta \) satisfies.

We let now the dot be the derivative w.r.t. the param. of the curve \( \eta \), \( \lambda \in V \):

\[ \gamma_0(\lambda) = \lambda \Rightarrow \gamma_0'(\lambda) = 1 \Rightarrow \eta_0'(\lambda) = 0 \]

From \( \Box \) we get:

\[ (\partial_t^2 \gamma)(t) = -(V' \circ \gamma)(t) = -(\partial_1 V \circ \gamma)(\gamma(t)) \]

\[ \Rightarrow \dot{\gamma}(t) = (\partial_1 \gamma)(\gamma(t)) \]

But \( \gamma_0'(\lambda) = 1 \), so we may rewrite this as:

\[ \dot{\gamma}(t) + \left[ (\partial_1 V \circ \gamma)(\gamma(t)) \right] \gamma_0' \gamma_0'' = 0 \]

We then recall what the geodesic eqn for a curve \( \eta: V \to \mathbb{R}^3 \) would look like given Christoffel symbols.
\[ y_i + \Gamma_{i j k} \dot{y}_j \dot{y}_k = 0 \]

For the two equations to coincide, we may define

\[
\begin{align*}
\Gamma_{0 i j} &:= 0 \\
\Gamma_{i 00} &= \Theta_{i} \dot{y}
\end{align*}
\]

for the other three eqns., \( i=1,2,3 \).

All other components should be zero.

Such a def. for \( \Gamma \) is clearly symmetric, giving (ii).

The corresponding Riemann curvature tensor, given by eqn (2.15), in the script:

\[
R_{ij kl} = \Gamma_{ij l m} - \Gamma_{ij m l} + \Gamma_{mj i l} - \Gamma_{mj li} \quad (i, j, k, l) \text{indices}
\]

\[
\Gamma_{ij l} \Gamma_{k l m} - \Gamma_{ij k} \Gamma_{l m} = \Gamma_{ij k m} + \sum_{s=1}^{k} \Gamma_{i j s} \Gamma_{k m} - \Gamma_{i j k} \Gamma_{m s} - \sum_{s=1}^{k} \Gamma_{i j s} \Gamma_{k m}
\]

\[
= -\sum_{s=1}^{k} \Gamma_{i j s} \Gamma_{k m} = 0 \quad \text{if} \quad k \in \{1,2,3\}.
\]

Also \( \Gamma_{ij k} \dot{y} = 0 \quad \text{if} \quad k \in \{1,2,3\}.

\implies \quad R_{ij kl} = \Gamma_{ij l k} \quad \text{if} \quad k \in \{1,2,3\}.

\implies \quad R_{i00} = \Gamma_{i00,ij} = 0 \quad \text{if} \quad i,j \in \{1,2,3\},

R_{i00,0} = 0 \quad \text{if} \quad i,j \in \{1,2,3\}.

If \( \Gamma \) were to be associated with a Levi-Civita connection, we know it should obey the symmetries on pp. 29, namely:

\[
\begin{align*}
\int_{a}^{b} \dddot{\Gamma}_{ijkl} &= -\dddot{\Theta}_{ij} R_{ijkl} \\
\int_{a}^{b} \dddot{\Gamma}_{ij kl} &= \dddot{\Theta}_{kl} R_{ijkl}
\end{align*}
\]

\[
\dddot{\jmath}_{ijkl} = \dddot{\jmath}_{ij k l} \quad \text{if} \quad \Theta_{k} \dddot{\jmath}_{i k d} \dddot{\jmath}_{d} = -\dddot{\Theta}_{ij} R_{ijkl}
\]

\implies \quad \text{Not true unless} \quad \dddot{\Theta}_{k} \dddot{\jmath}_{k d} \dddot{\jmath}_{d} = 0 \iff \text{Const. grav. field.}
Let $M$ be a manifold.

Def.: A subset $N \subseteq M$ is called an embedded submanifold of $M$ iff $\forall x \in N \exists$ chart $\varphi: U \to \mathbb{R}^m$ (We Open($U$), $\varphi(x)$ st. $\varphi(U \cap N) = \varphi(U \cap \mathbb{R}^n)$ where $W \subseteq \mathbb{R}^m$ is a subspace of dim $m = \dim(N)$, $0 \leq n \leq m$. The atlas for $N$ is the pair $(U, \varphi)$.

$\iota: N \to M$, the inclusion map, induces a tangent maps
$\iota_*: T_pN \to T_pM$, which is injective as well: (We also use $T_pN \cong \mathbb{R}^n$)

$$\begin{align*}
(\iota_*)_p \, (0) &= (dx)(0) \\
\iota(0) &= \iota(0) \\
\iota(0) &= \iota(0) = \iota(0) \\
\iota(0) &= \iota(0) \forall \, f \in F(M) \\
\text{But since } \forall \, g \in F(N), \exists \text{ extension } \tilde{g} \in F(M) : \tilde{g} \circ \iota = g, \forall = N, \\
\Rightarrow \text{ We think } \iota_*: T_pN \text{ as a linear subspace of } T_pM.
\end{align*}$$

If $g \in C(T^*M \otimes T^*M)$ is a metric on $M$, a $\mathbb{R}$ metric $h \in \Gamma(T^*N \otimes T^*N)$ is induced from the eqn:

$$h(X, Y) := g((dx)(X), (dx)(Y))$$

$\forall \, (X, Y) \in \Gamma(TN)^2$.

$\forall \, p \in N$, define $\pi_p: T_pM \to T_pN$ as the orthogonal proj.

assoc. to $g$ is an inner prod. on $T_pM$, so it defines what it means for two vectors to be orthogonal:

$$\forall \, \langle X, Y \rangle = 0 \iff \left[ \begin{array}{c} X_p \\ Y_p \end{array} \right] = 0 \quad \forall \, (X, Y) \in \Gamma(T^2M)^2$$

Then the orthogonal projection $\pi_p$ satisfies:

$$\begin{align*}
\pi_p \circ \pi_p \, &\, = \pi_p, \quad \forall \, p \in M, \forall \, \iota \in T_pM. \quad \text{Bes. $\pi_p$ is a proj.} \\
\pi_p (\iota \circ (dx)(U)) &= \pi_p (\iota \circ \pi_p (dx)(U)) = \pi_p (\iota \circ (dx \circ U)) & \forall \, \xi \in T_pN, \iota \in T_pM.
\end{align*}$$

Bes. it is an orthogonal proj.
Let $\nabla$ be the $\mathfrak{g}$-Levi-Civita connection.

\[
\mathcal{C}_{i,0}^{\mathfrak{g}} \quad \mathcal{C}^\rho \quad \nabla^\rho \quad \text{is an affine connection on } N.
\]

\[
\mathcal{P}_{f_{10}}^{\mathfrak{g}}(\mathcal{C}) \quad \text{W.r.t. } \quad \mathcal{P}^\rho \quad \nabla^\rho Y \quad \text{is } \mathbb{R}\text{-lin. in } X.
\]

\[
\mathcal{P}^\rho \quad \nabla^\rho Y \quad \text{is an affine con.} \quad \mathcal{P}^\rho = \mathcal{P}^\rho_1 \quad \nabla^\rho X + \mathcal{P}^\rho_2 \quad \nabla^\rho Y
\]

\[
(\mathcal{P}^\rho \quad \nabla^\rho X \quad \nabla^\rho Y)_p = \mathcal{P}^\rho \quad \nabla^\rho X \quad \nabla^\rho Y
\]

\[
\mathcal{P}^\rho \quad \nabla^\rho X \quad \text{is } \mathbb{R}\text{-lin. in } X.
\]

\[
\mathcal{P}^\rho \quad \nabla^\rho Y \quad \text{is } \mathbb{R}\text{-lin. in } Y.
\]

\[\mathcal{P}^\rho \quad \nabla^\rho X \quad \nabla^\rho Y \quad \text{obeys Leibniz in } X:
\]

\[
(\mathcal{P}^\rho \quad \nabla^\rho X \quad \nabla^\rho fY)_p = \mathcal{P}^\rho \quad (\nabla^\rho X \quad f \quad \nabla^\rho Y)_p
\]

\[
\mathcal{P}^\rho \quad \nabla^\rho X \quad \text{is } \mathbb{R}\text{-lin. in } X.
\]

\[
f \text{ extends } \quad \mathcal{P}^\rho \quad \nabla^\rho X \quad \mathcal{P}^\rho \quad \nabla^\rho Y
\]
\[ \tilde{\mathbf{P}}_\sigma \tilde{\mathbf{P}} (\nabla_{\mathbf{P}_x} x_\lambda) \mathbf{P}_x + (\tilde{\mathbf{P}}_\sigma (x_\lambda) \mathbf{P}) (x_\lambda) (\mathcal{P}) \]

\[ \Rightarrow \tilde{\mathbf{P}}_\sigma \nabla_{\mathbf{P}_x} x_\lambda \] is an affine conn. on \( M \).

**Claim:** The torsion of \( \nabla \) is zero.

**Proof:** The torsion is given by:

\[ \tilde{\mathbf{P}} \nabla_{\mathbf{P}_x} x_\lambda - \tilde{\mathbf{P}} \nabla_{\mathbf{P}_x} x_\lambda x - [x_\lambda, y] = \]

\[ \tilde{\mathbf{P}} (\nabla_{\mathbf{P}_x} x_\lambda - \mathbf{P}_x y) \mathbf{P}_x x - [x_\lambda, y] \]

**Claim:** \( \mathbf{P}_x [x_\lambda, y] = [x_\lambda x, \mathbf{P}_x y] \)

**Proof:**

\[ (\mathbf{P}_x [x_\lambda, y]) (\mathbf{P}) = ([x_\lambda, y]) (\mathbf{P}_x) \]

\[ = x_\lambda (\mathbf{P}_x y(\mathbf{P}_x )) - \mathbf{P}_x y(x_\lambda (\mathbf{P}_x )) \]

\[ = [x_\lambda x, \mathbf{P}_x y] \]

\[ (\mathbf{P}_x [x_\lambda, y]) (\mathbf{P}_x) = ([x_\lambda x, \mathbf{P}_x y]) (\mathbf{P}_x) \]

\[ = [x_\lambda x, \mathbf{P}_x y(\mathbf{P}_x )) - \mathbf{P}_x y([x_\lambda x, \mathbf{P}_x y]) \]

\[ = x_\lambda (\mathbf{P}_x y(\mathbf{P}_x )) - \mathbf{P}_x y([x_\lambda x, \mathbf{P}_x y]) \]

\[ = [x_\lambda x, \mathbf{P}_x y] \]

\[ \Rightarrow \tilde{\mathbf{P}}_\sigma \nabla x x \] as \( \nabla \) has no torsion.

**Claim:** \( (\tilde{\mathbf{P}}_\sigma \nabla x) (\mathcal{P}) = 0 \)

**Proof:** We use the inductive formula:

\[ (\nabla_{\mathbf{P}_x} x) (x_1, \ldots, x_k, \mu_1, \ldots, \mu_r) = \nabla_{\mathbf{P}_x} (T(x_1, \ldots, x_k, \mu_1, \ldots, \mu_r)) - T(\nabla_{\mathbf{P}_x} x_1, \ldots, \nabla_{\mathbf{P}_x} x_k, \mu_1, \ldots, \mu_r) \]

\[ = T(\nabla_{\mathbf{P}_x} x_1, \ldots, \nabla_{\mathbf{P}_x} x_k, \mu_1, \ldots, \mu_r) \]

\[ = T(\mathbf{P}_x x, x_1, \ldots, x_k, \mu_1, \ldots, \mu_r) \]

\[ = T(x_1, \ldots, x_k, \mu_1, \ldots, \mu_r, \mathbf{P}_x x) \]

\[ \Rightarrow ([x_\lambda x, \mathbf{P}_x y] = \mathbf{P}_x [x_\lambda, y] \)
Hence by uniqueness of the limit, we obtain the result.