GR – HW #6 — 25/10/2017

Q1 An Ideal Fluid

Ideal fluid with particles of mass m.
In rest frame # density n.
Isotropically distributed velocities of fixed length $u \in \mathbb{R}_{>0}$:
The velocity $v \in \mathbb{R}^3$ is equal $v = ue$ $\exists$ $e \in S^2$. e should be uniformly distributed.

The energy density ε is given by n times the energy of a single particle, which by SR we know is given by [redacted] with $\gamma \equiv (1-v^2)^{-1/2}$ the Lorentz factor:

$$\varepsilon = \text{[redacted]}$$

The pressure is obtained via the relation with the velocity:

(pressure) $\quad p =$ [redacted] $= \frac{1}{3}$ [redacted]

We saw in the lecture (eq-n (5.4)) that

$$\boxed{T = (\varepsilon + p) u^b \otimes u^b - p g}$$ metric

where u is the 4-velocity of the local rest frame (not of constituent particles).

i) For $v \to 0$: $\gamma \to 1$, $\varepsilon \to$ [redacted] and $p \to$ [redacted] so $\boxed{T \to \text{[redacted]}}$.
Recall that was T's form for dust.

ii) $v \to 1$, $m \to 0$, $\gamma m \to E$ (photons): $\varepsilon \to$ [redacted], $p \to$ [redacted],

so $\boxed{T \to \text{[redacted]}}$

The trace of a [redacted], such as T, is defined by first letting the metric "convert" it into a (1,1)-tensor:

$$T \rightsquigarrow T(g^{-1}(\cdot, -), \cdot)$$

Then the trace of a (1,1)-tensor is def. on simple tensors $w \otimes v$ as $tr(w \otimes v) := w(v)$ and extended linearly. $\Rightarrow$ $tr(u^b \otimes u^b) = g(u,u) = 1$ as this is a Lorentz scalar which is 1 in the local rest frame

$\boxed{2}$ $\mathrm{tr}(g(g^{-1}(\cdot,-),\cdot)) = \mathrm{tr}(\mathbb{1}_{TM\to T^*M}) = \dim(T_pM) = 4.$

$\Rightarrow \mathrm{tr}(T) =$ [redacted].

For electromagnetism, the field is given by the 2-form $F \in \Gamma(T^*M \otimes T^*M)$. Then T of F is

$$T_{\mu\nu} = F_{\mu\sigma} F_{\sigma'\nu}\, g^{\sigma'\sigma} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma}\, g_{\mu\nu}$$

$\Rightarrow \mathrm{tr}(T) =$ [redacted]

$\rightsquigarrow$ Photon gas is like radiation field.

A Variational Principle

Charged particle with trajectory $\gamma : \mathbb{R} \to \mathcal{M}$
$\tau \mapsto \gamma(\tau)$

Ambient EM field given by the 1-form $A \in \Gamma(T^*\mathcal{M})$
$F \equiv dA$

Now the E.o.M is :

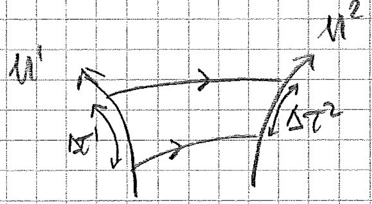
$$\nabla_{\dot\gamma} \dot\gamma = F(\cdot, \dot\gamma)$$

We want to verify that this E.o.M. comes from the Lagrangian $L(\gamma, \dot\gamma) = \sqrt{g(\dot\gamma, \dot\gamma)} + A(\dot\gamma)$

Note that to get the Euler-Lagrange eq-ns we must assume the end points do not vary, with no constraints on the parametrization. Thus we should let γ be param. arbitrarily as we make variations rather than with unit velocity

Q3 Another Look @ the Gravitational Redshift

(i) $\gamma : \lambda \mapsto M$ geodesic connecting sender (1) and receiver of 4-velocities u^1, u^2 respectively, representing the worldline of a light signal.



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Cl.: $$\frac{\nu_2}{\nu_1} = \frac{g_2(\dot{\gamma}, u)}{g_1(\dot{\gamma}, u)}$$

Pf.: $$\frac{\nu_2}{\nu_1} \equiv \frac{\Delta\tau^{(1)}}{\Delta\tau^{(2)}}$$ for monochromatic light.

$$\delta x^i = u^i \delta\tau^i$$

for null geodesics

$$L = \frac{1}{2} g(\dot{x}, \dot{x}) = \text{[redacted]}$$

But also $\int_1^2 L \, d\lambda =$ [redacted]

$\Rightarrow$ [redacted] $\Leftrightarrow$ [redacted]

ii) Assume g is time indep.

$\Rightarrow$ $\varphi_{t_0} : (t, x) \mapsto (t + t_0, x)$ is an isometry.

Its generator $K = \partial_t$ is a Killing vector.

By Noether's thm., $g(K, \dot{x})$ is const along geodesics.

Note u and K are parallel for fixed positions.

Hence [redacted]

$g(u,u) = 1$ bcs. particle geodesic,
$g(K,K) = g_{00}$

$\Rightarrow \frac{\nu_2}{\nu_1} =$ [redacted]

(last cancellation by Noether again.)