GR – HW #7 — 2/11/2017

Q1 Charged Dust

Dust of particles of mass m and charge e.

Let $\rho : \mathcal{M} \to \mathbb{R}$ be the mass density in a local frame

$u \in \Gamma(T\mathcal{M})$ velocity field

$F \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ EM field.

(i) Cl: $tr(\nabla j) = 0$ without reference to Maxwell's eq-ns,

where $j \equiv$ current density of charge

Pf: Recall by (5.2) we found [redacted] for dust where ρ is the mass density. But

$j =$ [redacted]

(ii) Cl: $tr(\nabla(T_{em} + T_{dust})) = 0$

Pf: Recall by Special Relativity, $tr(\nabla T_{em}) =$ [redacted]

(4.12), (4.14)

For dust, we have

$T_d = \rho\, u \otimes u$

pp. 18

$\Rightarrow \quad tr(\nabla(T_d)) =$ [redacted] $=$ [redacted]

$tr(\nabla(\rho u)) = 0$

$=$ [redacted] $=$ [redacted]

$=$ [redacted] $=$ [redacted] $=$ [redacted]

(4.15)

2. (b)?

On Conservation Laws

Goal: See how infinitesimal conservation eq-ns on j and T lead to global conserved quantities as one goes from SR → GR.

$\mathrm{tr}(\nabla j) = 0$ leads to charge conservation ✓ (as we shall see promptly)

$\mathrm{tr}(\nabla T) = 0$ does <u>not</u>.

Def.: A 3-dim submanifold of $\mathcal{M}$, Σ, is called spacelike iff $\forall\ (p,q) \in \Sigma^2$, p and q are spacelike separated, that is $g(p,q) > 0$.

Gauss' Theorem: pp. 15 $\displaystyle\int_{\mathcal{M}} \underbrace{(\mathrm{div}_\eta X)\eta}_{\parallel} = \int_{\partial\mathcal{M}} i_X \eta$

η is the volume n-form $\quad d(i_X\eta) = \mathcal{L}_X \eta$

Let W be a vector field

Let $\varphi : U \to \mathbb{R}^n$ be a chart

Cl.: $\mathrm{tr}(\nabla W)\sqrt{-\det(g^\varphi)} = \partial_i \sqrt{-\det(+g^\varphi)}\, W_i^\varphi$

Pf.: [redacted]

By Jacobi's formula, $\partial_i \det(g^\varphi) = \det(g^\varphi)\,\mathrm{tr}(g^{\varphi^{-1}} \partial_i g^\varphi)$

$\Rightarrow$ [redacted]

ii) Then we get, for $D \subseteq M$ bdd. domain

$$\int_D \mathrm{tr}(\nabla W)\sqrt{-\det(g_{ij})} =$$ [redacted]

Gauss on $\mathbb{R}^4$, n_i normal to ∂D: $\overset{\text{Gauss}}{=} \int_{\partial D}$ [redacted]

In appropriate choice of coordinates where

$$g = \begin{bmatrix} g_{00} & 0 \\ 0 & g_{\partial D} \end{bmatrix}$$

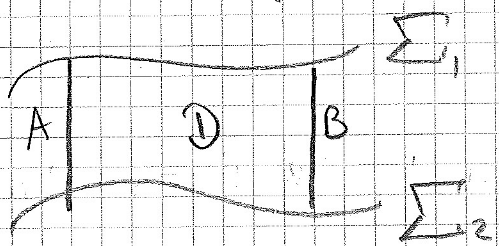
We find $\int_{\partial D}$ [redacted] sign dependent on where $g(n,n) = \pm 1$.

The total charge on Σ is given by:

$$Q(\Sigma) := \int_\Sigma g(j,n)\sqrt{-g_\Sigma}$$

Cl.: $Q(\Sigma)$ is indep. of Σ.

Pf.:



D regularizing cutoff bounded domain between them.

$$Q(\Sigma_1) - Q(\Sigma_2) =$$ [redacted]

$\overset{\text{Gauss}}{=} \int_D$ [redacted] $- \int_A - \int_B$ [redacted]

$j \to 0$ at ∞

$\boxed{4}$ ii) For a $(0,2)$ tensor, $tr(\nabla T)_{;j} \sqrt{-\det(g^{ij})} \neq \partial_i \sqrt{-\det(g^{ij})}\, T_{ij}$

Hence this fails!

Indeed, we find

$$\boxed{\sqrt{-\det(g^{ij})}\; tr(\nabla T)_{;j} = \text{[redacted]}}$$