GR – HW #8 – 8/11/2017

Q1 Energy Conditions

Let $\mathcal{M}$ be a 4-dim pseudo-Riem. mnfld. w/ metric g of signature $+1,-1,-1,-1$.
Let $p\in\mathcal{M}$ be given, and $\{e_\alpha\}_\alpha$ a basis of $T_p\mathcal{M}$ with $g_p(e_\alpha, e_\beta)=\eta_{\alpha\beta}$ with $\eta=\mathrm{diag}(1,-1,-1,-1)$.
The 4-velocity of an observer at rest is e_0.

i) Def.: The cone of future-oriented, timelike or light-like vectors is $\overline{V^+} := \{ v\in T_p\mathcal{M} \mid g_p(v,v)\geq 0 \wedge v_0\geq 0\}$

where $v_0\in\mathbb{R}$ is the component of v in $\{e_\alpha\}_\alpha$: $v_0 = v(e_0^*)$.

Cl.: $\forall\, v\in T_p\mathcal{M}, \left\{ v\in\overline{V^+} \Leftrightarrow \left[g_p(v,w)\geq 0 \;\forall\, w\in\overline{V^+}\right]\right\}$

Pf.: $\boxed{\Rightarrow}$ Let $v\in\overline{V^+}$, $w\in\overline{V^+}$. Want $g_p(v,w)\geq 0$.

Know $g_p(v,v)\geq 0$, $v_0\geq 0$, $g_p(w,w)\geq 0$, $w_0\geq 0$.

$g_p(v,w)\geq 0 \Leftrightarrow v_0w_0 \geq v_1w_1+v_2w_2+v_3w_3$

By Cauchy-Schwarz, $v_1w_1+v_2w_2+v_3w_3 \leq (v_1^2+v_2^2+v_3^2)^{1/2}\times$
$\times(w_1^2+w_2^2+w_3^2)^{1/2}$
$\leq (v_0^2)^{1/2}(w_0^2)^{1/2} \underset{\substack{\uparrow \\ v_0\geq 0,\, w_0\geq 0}}{\leq} v_0w_0$

$\boxed{\Leftarrow}$ Let $v\in T_p\mathcal{M}$: $g_p(v,w)\geq 0 \;\forall\, w\in\overline{V^+}$. Want $g_p(v,v)\geq 0 \wedge v_0\geq 0$.

Define $w\in T_p\mathcal{M}$ with components $\left(1, \frac{v_i}{(v_1^2+v_2^2+v_3^2)^{1/2}}\right)$.

Cl.: $w\in\overline{V^+}$

Pf.: $w_0 \equiv 1 > 0$ ✓

$g_p(w,w) = 1-1 = 0 \geq 0$ ✓

$\Rightarrow g_p(v,w)\geq 0$

But $g_p(v,w) = v_0 - (v_1^2+v_2^2+v_3^2)^{1/2}$

$\Rightarrow v_0\geq 0$ ✓ and $v_0^2 - v_1^2 - v_2^2 - v_3^2\geq 0$ ✓

ii) Let $(a,b)\in\mathbb{R}^2$. $u\in T_p\mathcal{M}$: $g_p(u,u)>0$ (timelike).

Define $T := a\, u^b\otimes u^b - b\, g_p(u,u)\, g$.

Cl.: $\left[\forall\, v\in T_p\mathcal{M},\; g_p(v,v)\geq 0 \Rightarrow g_p(v, (Tv)^\#)\geq 0\right] \Leftrightarrow a>0 \wedge b\leq a$.

Let $v \in T_pM$ be given s.t. $g_p(v,v) \geq 0$.

$$Tv = a \underbrace{u^\flat(v)}_{g_p(u,v)} u^\flat - b g_p(u,u) g_p(v,\cdot)$$

$$(Tv)^\# = a g_p(u,v) u - b g_p(u,u) v$$

$$\Rightarrow g_p(v,(Tv)^\#) = a g_p(u,v) g_p(v,u) - b g_p(u,u) g_p(v,v)$$
$$= a(g_p(u,v))^2 - b g_p(u,u) g_p(v,v)$$
$$= a((g_p(u,v))^2 - g_p(u,u) g_p(v,v)) + (a-b) \underbrace{g_p(u,u)}_{\geq 0} \underbrace{g_p(v,v)}_{\geq 0}$$

Cl.: $g_p(u,v)^2 \geq g_p(u,u) g_p(v,v)$

Pf.: WLOG we may assume $\{e_\alpha\}_\alpha$ has been chosen s.t. $u = u_0 e_0$ $\exists u_0 \in \mathbb{R}$.
Then $g_p(u,v)^2 = u_0^2 v_0^2$, $g_p(u,u) = u_0^2$,

$$\Rightarrow \cancel{u_0^2} v_0^2 \overset{?}{\geq} \cancel{u_0^2} (v_0^2 - \|\vec{v}\|^2)$$

$$0 \geq -\|\vec{v}\|^2 \quad \checkmark$$

Since these two terms may be made to equal zero independently of one another, we find the result.

iii) Cl.: $T_{00} \geq 0$ in every basis (for some $(0,2)$-tensor T) iff $\forall v \in T_pM : g_p(v,v) \geq 0$, $\underbrace{g_p(v,(Tv)^\#)}_{= T(v,v)} \geq 0$

Pf.: Since the 4-velocity u of an observer at rest is e_0 in a basis $\{e_\alpha\}_\alpha$, $T_{00} = g_p(u,(Tu)^\#)$.

To generalize from $u = e_0$, we simply need to pick any u with $g_p(u,u) \geq 0$.

iv) Let T be a $(0,2)$-tensor.

Cl.: $T_{00} + \sum_{i=1}^{3}{}' T_{ii} \geq 0$ in every basis iff $T(v,v) - \frac{1}{2} \operatorname{tr}(T) g(v,v) \geq 0$ for every $v \in T_pM : g_p(v,v) \geq 0$.

Pf.: Recall that since T is a $(0,2)$-tensor, its trace

$$\left.\begin{array}{l} g_p(v,(Tv)^\#) = T(v,v) \geq 0 \\ g_p((Tv)^\#,(Tv)^\#) \geq 0 \end{array}\right\} \forall\ v \in T_p\mathcal{M}:\ g_p(v,v) \geq 0$$

Pf.: The 1^{st} condition has already been demonstrated in (iii).

The 2^{nd} condition is
$g_p(T(e_0,\cdot),T(e_0,\cdot)) = g_p((Te_0)^\#,(Te_0)^\#)$.

Note: The above cond. is also equivalent to $T(e_0,\cdot) \in \overline{V}^+$.

Cl.: The above cond. is also equivalent to $T_{00} \geq |T_{\alpha\beta}|$ $\forall \alpha,\beta$, in any basis.

Pf.: Let $v \in T_p\mathcal{M}$ be given with $g_p(v,v)$.

Then we assume $T(v,\cdot) \in \overline{V}^+$.

By (i), we know that equivalent to $g_p(T(v,\cdot)^\#,w) \geq 0$ $\forall w \in \overline{V}^+$.

$\boxed{\Rightarrow}$ Pick $v_0 = w_0 = 1$ (so $\|\vec{v}\| \leq 1$, $\|\vec{w}\| \leq 1$). Then

$g_p(T(v,\cdot)^\#,w) = T(v,w) = T_{\alpha\beta}v_\alpha w_\beta \geq 0$ +

With $\vec{v} = \vec{w} = 0$, we get $T_{00} \geq 0$, which is (iii).

with $v = \pm e_i$, $w = 0$ we get $T_{00} \pm T_{0i} \geq 0 \Leftrightarrow T_{00} \geq |T_{0i}|$.

with $v = \pm e_i, w = \pm e_k$ we get $T_{00} \pm (T_{0i} + T_{0k}) + T_{ik} \geq 0$.

$\overset{+}{\Rightarrow}$ $T_{00} + T_{ik} \geq 0$ and $T_{00} - T_{ik} \geq 0$.

$\boxed{\Leftarrow}$ We may perform Euclidean on $\vec{e}$ to get $T_{01} = T_{02} = 0$ WLOG. Then $T_{00} \geq |T_{03}|$ implies $T_{00} \geq 0$ and $T_{0\cdot}$ is timelike ✓

vi) Perfect Fluid $T \equiv (\varepsilon+p)u^\flat \otimes u^\flat - p\, \overset{=1}{g_p(u,u)}\, g$ $\quad tr(T) = \varepsilon + p - 4p = \varepsilon - 3p$

$\longrightarrow$ T as in (ii) with $a = \varepsilon + p$, $b = p$ (p pressure, ε energy density).

⊛ (iii) implies (by (ii)) that $\boxed{\varepsilon + p \geq 0}$ and $p \leq \varepsilon + p$

$\Updownarrow$

$\boxed{\varepsilon \geq 0}$

$\longrightarrow$ $T - \frac{1}{2}tr(T)g = (\varepsilon+p)u^\flat \otimes u^\flat - pg - \frac{1}{2}(\varepsilon - 3p)g$

$= (\varepsilon+p)u^\flat \otimes u^\flat - \frac{1}{2}(\varepsilon - p)g$ as in (ii) with $a = \varepsilon + p$, $b = \frac{1}{2}(\varepsilon - p)$

⊛ (iv) implies (by (ii)) that $\varepsilon + p \geq 0$ and $\varepsilon + p - \frac{1}{2}(\varepsilon - p) \geq 0$ $\Leftrightarrow$ $\boxed{\varepsilon + 3p \geq 0}$

needs to be defined by first converting one of its slots to a covector using the metric.

In a basis $\{e_\alpha\}_\alpha$ as above,

$$T = T(e_\alpha, e_\beta)\, e_\alpha^* \otimes e_\beta^* \quad \text{where } e_\alpha^* \equiv e_\alpha^{\flat}$$

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$$\Rightarrow \quad T(g_p(\cdot,-),\cdot) = T(e_\alpha, e_\beta)\, e_\alpha \otimes e_\beta^*$$

$$tr(T) = T(e_\alpha, e_\beta)\, tr(e_\alpha \otimes e_\beta^*) = T(e_\alpha, e_\beta)\, g_p(e_\alpha, e_\beta)$$
$$= T_{00} - T_{11} - T_{22} - T_{33}$$

OTOH, $T(e_0, e_0) - \frac{1}{2} tr(T)\, g(e_0, e_0) =$
$$= T_{00} - \tfrac{1}{2}(T_{00} - T_{11} - T_{22} - T_{33})\, g_{00}$$
$$\overset{g_{00}=1}{=} \tfrac{1}{2}(T_{00} + T_{11} + T_{22} + T_{33})$$

Again the way to generalize from a particular basis with e_0 is to take any timelike vector $v \in T_pM$.

<u>Cl.:</u> Let T be the $(0,2)$-energy momentum tensor. Then

$$\left[T(v,v) - \tfrac{1}{2} tr(T)\, g(v,v) \geq 0 \quad \forall\, v \in T_pM : g_p(v,v) \geq 0 \right]$$

$$\Leftrightarrow$$

$$Ricci(v,v) \geq 0$$

<u>Pf.:</u> Recall the Einstein field eq-ns: the

$$\boxed{G = 8\pi T} \quad ⊛$$

$$\rho(X,Y) \equiv \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

where G is the Einstein $(0,2)$-tensor:

$$tr(Z \mapsto \rho(Z,X)Y) \equiv Ricci(X,Y)$$

$$G \equiv Ricci - \tfrac{1}{2} R\, g \quad , \quad R \equiv tr(Ricci)$$

Take the trace of ⊛ to get:

$$tr(G) = \underbrace{tr(Ricci)}_{\equiv R} - \tfrac{1}{2} R\, \underbrace{tr(g)}_{=4} = -R$$

$$\Rightarrow \quad -R = -8\pi\, tr(T)$$

$$\Rightarrow \quad Ricci = 8\pi T + \tfrac{1}{2} R\, g = 8\pi T - 4\pi\, tr(T)\, g$$

$$\boxed{Ricci = 8\pi \left(T - \tfrac{1}{2} tr(T)\, g\right)}$$

Alternative formulation of E.F.E.

1b) <u>Cl.:</u> If T is a $(0,2)$-tensor then the condition $T_{00} \geq 0$ and $g_p(T(e_0,\cdot), T(e_0,\cdot)) \geq 0$ in every basis $\{e_\alpha\}_\alpha$ is equivalent to

$$\rightarrow \quad g((Tu)^\sharp, (Tu)^\sharp) = u_\alpha T_{\alpha\beta}\, g_{\beta\gamma}\, T_{\gamma\delta}\, u_\delta\, g_{\alpha\delta}$$
$$= g(u, (``T^T T" u)^\sharp)$$

But T is symmetric, so $T^T T = T^2$.

$$(T^2)_{\alpha\beta} \equiv T_{\alpha\gamma}\, g_{\gamma\delta}\, T_{\delta\beta}$$

If
$$T = a u^\flat \otimes u^\flat - b\, g(u,u) g$$
$$T(g^{-1}(\cdot,-),\cdot) = a u \otimes u^\flat - b\, g(u,u) \mathbb{1}$$
$$T^2 = a^2 u^\flat \otimes u^\flat g(u,u) - ab\, g(u,u)\, u^\flat \otimes u^\flat - b\, g(u,u)\, a\, u^\flat \otimes u^\flat + b^2 g(u,u)^2 g$$
$$= g(u,u)\, a(a-2b)\, u^\flat \otimes u^\flat - (-b^2 g(u,u)^2) g$$

⊛ (iv) implies (by (ii)) that $\cancel{g(u,u)}\, \underbrace{a(a-2b)}_{(\varepsilon+p)\underbrace{(\varepsilon+p-2p)}_{\varepsilon-p}} > 0$

$$\underbrace{\qquad}_{\varepsilon^2 - p^2}$$

Since $\varepsilon \geq 0$, this implies $\boxed{\varepsilon \geq |p|}$

The other condition from (ii) is $g(u,u)(\varepsilon^2 - p^2) + b^2 g(u,u)^2 \geq 0$ which is always true. ✓

Electromagnetism

$T \equiv$ something

Note $\mathrm{tr}(T) = 0$ (HW6Q1) so both (iii) and (iv) imply the same condition, $T_{00} \geq 0$.
We know for EM, $T_{00} = \frac{1}{2}(\|\vec{E}\|^2 + \|\vec{B}\|^2)$, so this is indeed satisfied.

(iv) is also satisfied:

$$T_{0i} = (\vec{E} \wedge \vec{B})_i \quad \text{and} \quad \|\vec{E} \wedge \vec{B}\| \leq \|\vec{E}\|\,\|\vec{B}\| \leq T_{00} \quad ✓$$

The Vacuum with the Cosmological Term

$T = \Lambda g$, $\Lambda > 0$

$T_{00} = \Lambda > 0$ ✓ $\Rightarrow$ (iii) is satisfied.

$T - \frac{1}{2}\mathrm{tr}(T) g = \Lambda g - 2\Lambda g = -\Lambda g$ does not have its 00 component positive! $\Rightarrow$ (iv) is not satisfied!

$T^2 = \Lambda^2 g \Rightarrow$ (v) is satisfied. ✓

Q2 Bound on the Cosmological Const

E.F.E.: $\boxed{G = 8\pi T}$ (5.11)

with a cosmological const, we get:

$$\boxed{G = 8\pi T + \Lambda g}$$

(i) $G \equiv \mathrm{Ricci} - \frac{1}{2} R g$

$\Rightarrow \mathrm{tr}(G) = -R \Rightarrow -R = 8\pi\, \mathrm{tr}(T) + 4\Lambda$

$\Rightarrow \mathrm{Ricci} + 4\pi\, \mathrm{tr}(T) g + 2\Lambda g = 8\pi T + \Lambda g$

$$\boxed{\mathrm{Ricci} = 8\pi\left(T - \tfrac{1}{2}\mathrm{tr}(T) g\right) - \Lambda g}$$

We saw in (5.16) that $(\mathrm{Ricci})(0,0) \sim \Delta\varphi$, φ being the grav. pot.

from the E.F.E. we get

$$(\mathrm{Ricci})_{00} = 8\pi\Big(\underbrace{T_{00}}_{\approx \rho} - \tfrac{1}{2}\underbrace{\mathrm{tr}(T)}_{\approx \rho \text{ mass density}} \overset{=1}{g_{00}}\Big) - \Lambda \overset{=1}{g_{00}}$$

$$= 4\pi\rho - \Lambda$$

$$\Rightarrow \boxed{\Delta\varphi = 4\pi\rho - \Lambda}$$

ii) For a point mass $\rho = M\delta$ (at the origin).

Then one can verify it is solved by

$$\varphi(x) = -\frac{M}{\|x\|} - \frac{1}{6}\Lambda \|x\|^2 \qquad (\text{as } \Delta\|x\|^2 = 6)$$

iii) $-(\nabla\varphi)(x) = -M\frac{x}{\|x\|^3} + \frac{1}{3}\Lambda x$ is the gravitational accel.

Additional term negligible as long as

$$M\|x\|^{-2} \gg \tfrac{1}{3}\Lambda\|x\|$$

Estimate with M = sun mass, $\|x\|$ = Pluto's orbit radius

$$\Leftrightarrow \boxed{\Lambda \ll 3M\|x\|^{-3}}$$