GR — HW9 — Solution

Q1 Expanding or Static Universe

Let the Minkowski metric be defined on the manifold $\mathbb{R}^5$ (yes, 5). With the chart $\psi: \mathbb{R}^5 \to \mathbb{R}^5$ given by the identity map the components of the Minkowski metric are given by

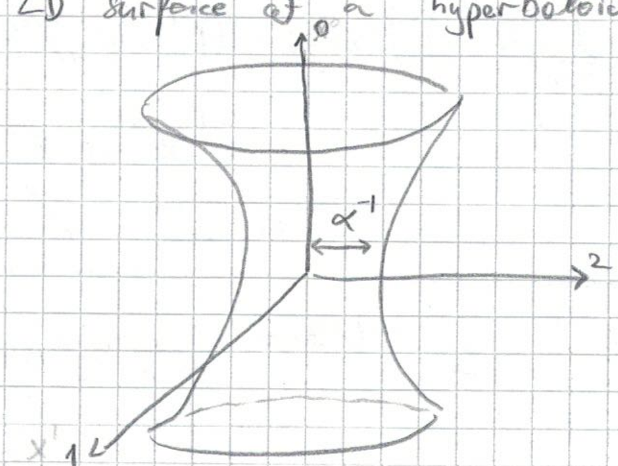
$$g^\psi = \mathrm{diag}(+1, -1, -1, -1, -1)$$

Define $\mathcal{M} := \left\{ x \in \mathbb{R}^5 \;\middle|\; -(x_0)^2 + \sum_{j=1}^{4} (x_j)^2 = \alpha^{-2} \right\} \quad \exists\, \alpha > 0.$

Note $\mathcal{M}$ is the preimage of $\{0\}$ of a smooth map and so it is a 4D smooth manifold — a smooth submanifold of $\mathbb{R}^5$.

How to picture $\mathcal{M}$?

If we forget about two dimensions for a moment, we can picture $\mathcal{M}$ as the 2D surface of a hyperboloid in $\mathbb{R}^3$;



The metric g on $\mathbb{R}^5$ induces a metric h on $\mathcal{M}$ by restriction (see HW5Q3).

i) Define $\mathcal{U} \in \mathrm{Open}(\mathcal{M})$ by $\mathcal{U} := \{ x \in \mathcal{M} \mid x_0 + x_1 > 0 \}$. Note that $\mathcal{U}$ is an int. of an open subset of $\mathbb{R}^5$ with $\mathcal{M}$ and is hence open in $\mathcal{M}$ by def. of the subsp. top.

Define a chart $\varphi^{-1}: \mathbb{R}^4 \to \mathcal{U}$ via:

hyperbolic sinus

$$\varphi^{-1}_0(t, y_1, y_2, y_3) := \alpha^{-1}\left(sh(\alpha t) + \frac{r^2}{2} e^{\alpha t}\right), \quad r^2 := \sum_{i=1}^{3} (y_i)^2.$$

$$\varphi^{-1}_1(t,y_1,y_2,y_3) := \alpha^{-1}\Big(\underset{\text{cosine hyperbolic}}{\mathrm{ch}(\alpha t)} - \frac{r^2 e^{\alpha t}}{2}\Big)$$

$$\varphi^{-1}_{j+1}(t,y_1,y_2,y_3) := \alpha^{-1} e^{\alpha t} y_j \quad \forall\ j \in \{1,2,3\}$$

for all $(t,y_1,y_2,y_3) \in \mathbb{R}^4$

<u>Cl.:</u> φ^{-1} is well-defined.

<u>Pf.:</u> We must show that $\forall\ (t,y_1,y_2,y_3) \in \mathbb{R}^4$, (abrev. as (t,y) below) we have $\varphi^{-1}(t,y) \in \mathcal{U}$.

First note $(\varphi^{-1}_0 + \varphi^{-1}_1)(t,y) = \alpha^{-1}(\underbrace{\mathrm{sh}(\alpha t) + \mathrm{ch}(\alpha t)}_{= \exp(\alpha t)}) = \dfrac{e^{\alpha t}}{\alpha} > 0$

$$(\varphi^{-1}_1 - \varphi^{-1}_0)(t,y) = \alpha^{-1}(\underbrace{(\mathrm{ch}-\mathrm{sh})(\alpha t)}_{e^{-\alpha t}} - r^2 e^{\alpha t}) = \alpha^{-1}(e^{-\alpha t} - r^2 e^{\alpha t})$$

$$\sum_{j=2}^{4} \varphi^{-1}_j(t,y) = \alpha^{-1} e^{\alpha t} \sum_{j=1}^{3} (y_j)^2 = \alpha^{-1} e^{\alpha t} r^2$$

$$\Rightarrow \Big((-\varphi^{-1}_0)^2 + \sum_{j=1}^{4} (\varphi^{-1}_j)^2\Big)(t,y) = \Big((\varphi^{-1}_1 - \varphi^{-1}_0)(\varphi^{-1}_1 + \varphi^{-1}_0) + \sum_{j=2}^{4} (\varphi^{-1}_j)^2\Big)(t,y)$$

$$= \alpha^{-1}(e^{-\alpha t} - r^2 e^{\alpha t})\alpha^{-1} e^{\alpha t} + (\alpha^{-1} e^{\alpha t})^2 r^2 =$$

$$= \alpha^{-2} \quad \checkmark$$

Define $\varphi : \mathcal{U} \to \mathbb{R}^4$ by $\mathcal{U} \ni x \mapsto \begin{bmatrix} \frac{1}{\alpha}\log(\alpha(x_0+x_1)) \\ \frac{x_2}{x_0+x_1} \\ \frac{x_3}{x_0+x_1} \\ \frac{x_4}{x_0+x_1} \end{bmatrix} \in \mathbb{R}^4$.

<u>Cl.:</u> $\varphi \circ \varphi^{-1} = \mathbb{1}_{\mathbb{R}^4}$ and $\varphi^{-1} \circ \varphi = \mathbb{1}_{\mathcal{U}}$

<u>Pf.:</u> $\varphi(\varphi^{-1}(t,y))_0 = \frac{1}{\alpha}\log\big(\alpha \frac{e^{\alpha t}}{\alpha}\big) = \frac{\alpha t}{\alpha} = t \quad \checkmark$

$\varphi(\varphi^{-1}(t,y))_j = \dfrac{\alpha^{-1} e^{\alpha t} y_j}{\alpha^{-1} e^{\alpha t}} = y_j \quad \checkmark \qquad (\forall\ j \in \{1,2,3\}.$

Note $\exp(\alpha \varphi(x)_0) = \exp(\alpha \frac{1}{\alpha} \log(\alpha(x_0+x_1)))$

$$= \alpha(x_0+x_1)$$

$$\exp(-\alpha\varphi(x)_0) = \exp(-\alpha \tfrac{1}{\alpha}\log(\alpha(x_0+x_1))) = \frac{1}{\alpha(x_0+x_1)}$$

$$\mathrm{sh}(\alpha\varphi(x)_0) = \frac{1}{2}\left(e^{\alpha\varphi(x)_0} - e^{-\alpha\varphi(x)_0}\right) = \frac{1}{2}\left(\alpha(x_0+x_1) - \frac{1}{\alpha(x_0+x_1)}\right)$$

$$\mathrm{ch}(\alpha\varphi(x)_0) = \frac{1}{2}\left(\alpha(x_0+x_1) + \frac{1}{\alpha(x_0+x_1)}\right)$$

$$\sum_{j=1}^{3}(\varphi(x)_j)^2 = \sum_{j=1}^{3}\left(\frac{x_{j+1}}{x_0+x_1}\right)^2$$

$$\varphi^{-1}(\varphi(x))_0 = \alpha^{-1}\left(\mathrm{sh}(\alpha\varphi(x)_0) + \tfrac{1}{2}(\circ)\, e^{\alpha\varphi(x)_0}\right)$$

$$= \alpha^{-1}\left(\frac{1}{2}\left(\alpha(x_0+x_1) - \frac{1}{\alpha(x_0+x_1)}\right) + \frac{1}{2}\left(\sum_{j=1}^{3}\frac{(x_{j+1})^2}{(x_0+x_1)^2}\right)\alpha(x_0+x_1)\right)$$

Note $x \in \mathcal{U} \subseteq \mathcal{M}$, so $-(x_0)^2 + \sum_{j=1}^{4}(x_j)^2 = \alpha^{-2}$.

$$\Longrightarrow \sum_{j=1}^{3}(x_{j+1})^2 = \alpha^{-2} + (x_0)^2 - (x_1)^2$$

$$= \alpha^{-2} + (x_0-x_1)(x_0+x_1)$$

So $\varphi^{-1}(\varphi(x))_0 = \frac{1}{2\cancel{\alpha^2}\cancel{(x_0+x_1)}}\left(\cancel{\alpha^2}(x_0+x_1)^{\cancel{2}} \cancel{-1} + \cancel{1} + \cancel{\alpha^2}(x_0-x_1)\cancel{(x_0+x_1)}\right)$

$$= x_0 \quad ✓$$

$$\varphi^{-1}(\varphi(x))_1 = \alpha^{-1}\left(\mathrm{ch}(\alpha\varphi(x)_0) - \frac{1}{2}\left(\sum_{j=1}^{3}\frac{(x_{j+1})^2}{(x_0+x_1)^2}\right)\alpha(x_0+x_1)\right)$$

$$= \frac{1}{2\cancel{\alpha^2}\cancel{(x_0+x_1)}}\left(\cancel{\alpha^2}(x_0+x_1)^{\cancel{2}} \cancel{+1} - \cancel{1} - (x_0-x_1)\cancel{(x_0+x_1)}\cancel{\alpha^2}\right)$$

$$= x_1 \quad ✓$$

$$\varphi^{-1}(\varphi(x))_j = \alpha^{-1}\alpha(x_0+x_1)\,\varphi(x)_{j-1} = (x_0+x_1)\frac{x_j}{x_0+x_1} = x_j \quad ✓$$

$$\forall j \in \{2,3,4\}.$$

$\Longrightarrow$ φ is a bijection. Note φ and φ^{-1} are smooth. Hence $\varphi: \mathcal{U} \to \mathbb{R}^4$ is a valid chart!

We want to express the induced metric h in the chart φ: h^φ

Let $\{\partial_i^\varphi\}_{i=0}^{3}$ be the chart basis of $T\mathcal{U}$. Then

$$h^\varphi \equiv \{h(\partial_i^\varphi, \partial_j^\varphi)\}_{i,j=0}^{3}$$

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We think of $\{d_j^\varphi\}_{j=0}^{3} \subseteq T\mathcal{U}$ as vectors in $T\mathbb{R}^5$ via $\mathcal{U}\subseteq\mathbb{R}^5$. To compute the induced metric h we thus need the components of each d_j^φ in the chart ψ:

$$\{(d_j^\varphi)^{\psi}_i\}_{i=0}^{4} \equiv \{d_j^\varphi(\psi_i)\}_{i=0}^{4}$$

$$d_j^\varphi(\psi_i) \equiv \left[\partial_j(\psi_i\circ\varphi^{-1})\right]\circ\varphi = \left(\partial_j({\varphi^{-1}}_i)\right)\circ\varphi$$

$$\begin{cases} (\partial_t {\varphi^{-1}}_0)(t,y) = \alpha^{-1}(ch(\alpha t)\alpha + \frac{1}{2}r^2e^{\alpha t}\alpha) = ch(\alpha t) + \frac{1}{2}r^2e^{\alpha t} \\ (\partial_t {\varphi^{-1}}_1)(t,y) = \alpha^{-1}(sh(\alpha t)\alpha - \frac{1}{2}r^2e^{\alpha t}\alpha) = sh(\alpha t) - \frac{1}{2}r^2e^{\alpha t} \\ (\partial_t {\varphi^{-1}}_{j+1})(t,y) = e^{\alpha t}y_j \quad \forall\, j\in\{1,2,3\} \\ (\partial_{y_j} {\varphi^{-1}}_0)(t,y) = \frac{1}{2\alpha}e^{\alpha t}2y_j = \frac{e^{\alpha t}}{\alpha}y_j \quad \forall\, j\in\{1,2,3\} \\ (\partial_{y_j} {\varphi^{-1}}_1)(t,y) = -\frac{1}{2\alpha}e^{\alpha t}2y_j = -\frac{e^{\alpha t}}{\alpha}y_j \quad \forall\, j\in\{1,2,3\} \\ (\partial_{y_j} {\varphi^{-1}}_{k+1})(t,y) = \frac{e^{\alpha t}}{\alpha}\delta_{j,k} \quad \forall\, j\in\{1,2,3\},\ k\in\{1,2,3\} \end{cases}$$

Now that we have these components we may compute h^φ: $\forall\,(i,j)\in\{0,1,2,3\}^2$

$$h^\varphi_{ij} \equiv h(d_i^\varphi, d_j^\varphi) \overset{TU\subseteq T\mathbb{R}^5}{=} g(d_i^\varphi, d_j^\varphi)$$

$$= \underbrace{g^\psi_{lm}}_{\text{Minkowski}}\, d_i^\varphi(\psi_l)\, d_j^\varphi(\psi_m) = d_i^\varphi(\psi_0)\,d_j^\varphi(\psi_0) - \sum_{l=1}^{4} d_i^\varphi(\psi_l)\,d_j^\varphi(\psi_l)$$

$$(h^\varphi_{00}\circ\varphi^{-1})(t,y) = (ch(\alpha t)+\tfrac{1}{2}r^2e^{\alpha t})^2 - (sh(\alpha t)-\tfrac{1}{2}r^2e^{\alpha t})^2 - e^{2\alpha t}r^2$$

$$= ch(\alpha t)^2 + ch(\alpha t)r^2e^{\alpha t} + \tfrac{1}{4}r^4e^{2\alpha t} - sh(\alpha t)^2 + sh(\alpha t)r^2e^{\alpha t}$$
$$- \tfrac{1}{4}r^2e^{2\alpha t} - e^{2\alpha t}r^2$$
$$= 1$$

$$(h^\varphi_{0j}\circ\varphi^{-1})(t,y) = (ch(\alpha t)+\tfrac{r^2}{2}e^{\alpha t})(\tfrac{e^{\alpha t}}{\alpha}y_j) - (sh(\alpha t)-\tfrac{1}{2}r^2e^{\alpha t})(-\tfrac{e^{\alpha t}}{\alpha}y_j)$$
$$-\sum_{l=1}^{3} e^{\alpha t}y_l\,\frac{e^{\alpha t}}{\alpha}\delta_{j,l}$$
$$= \frac{e^{2\alpha t}}{\alpha}y_j - \frac{e^{2\alpha t}}{\alpha}y_j = 0 \qquad \forall\, j\in\{1,2,3\}.$$

$$(h^\varphi_{jk}\circ\varphi^{-1})(t,y) = \cancel{\frac{e^{\alpha t}}{\alpha}y_j\,\frac{e^{\alpha t}}{\alpha}y_k + \frac{e^{\alpha t}}{\alpha}y_j\,\frac{e^{\alpha t}}{\alpha}y_k} - \left(\frac{e^{\alpha t}}{\alpha}\right)^2\sum_{l=1}^{3}\delta_{j,l}\delta_{k,l}$$

$$= -\frac{e^{2\alpha t}}{\alpha^2}\delta_{j,k} \qquad \forall\ (j,k) \in \{1,2,3\}^2$$

$$\Rightarrow \boxed{(h^{\varphi} \circ \varphi^{-1})(t,y) = \mathrm{diag}\left(1, -\frac{e^{2\alpha t}}{\alpha^2}, -\frac{e^{2\alpha t}}{\alpha^2}, -\frac{e^{2\alpha t}}{\alpha^2}\right)}$$

Note that this metric is precisely the spatially flat expanding solution!

ii) Define another chart $\eta^{-1}: \mathbb{R}\times(0,\alpha^{-1})\times(0,2\pi)^2 \longrightarrow \mathrm{im}(\eta^{-1}) \cap \mathcal{M}$ as follows: (in the following we omit the (t,r,θ,φ) argument)

$$\begin{cases} \eta^{-1}{}_0 := \alpha^{-1}\sqrt{1-\alpha^2 r^2}\,\mathrm{sh}(\alpha t) \\ \eta^{-1}{}_1 := \alpha^{-1}\sqrt{1-\alpha^2 r^2}\,\mathrm{ch}(\alpha t) \\ \eta^{-1}{}_2 := r\sin(\theta)\cos(\varphi) \\ \eta^{-1}{}_3 := r\sin(\theta)\sin(\varphi) \\ \eta^{-1}{}_4 := r\cos(\theta) \end{cases}$$

Cl.: η^{-1} is well-defined.

Pf.: $\eta^{-1}_0 + \eta^{-1}_1 = \alpha^{-1}\sqrt{1-\alpha^2r^2}\,e^{\alpha t}$

$\eta^{-1}_1 - \eta^{-1}_0 = \alpha^{-1}\sqrt{1-\alpha^2r^2}\,e^{-\alpha t}$

$$\Rightarrow -(\eta_0^{-1})^2 + \sum_{e=1}^{4}(\eta_e^{-1})^2 = \left(\alpha^{-1}\sqrt{1-\alpha^2r^2}\right)^2 + r^2 = \alpha^{-2} \checkmark$$

Note again $\eta_0^{-1} + \eta_1^{-1} > 0$, $\eta_1^{-1} - \eta^{-1}_0 > 0$, $\eta^{-1}_1 > 0$ and some more conditions. Define $V := \mathrm{im}(\eta^{-1}) \subseteq \mathcal{M}$.

Define $\eta: V \longrightarrow \mathbb{R}\times(0,\alpha^{-1})\times(0,2\pi)^2$ by:

$$\eta(x)_t := \mathrm{arctgh}\left(\frac{x_0}{x_1}\right) = \frac{1}{2}\log\left(\frac{1+\frac{x_0}{x_1}}{1-\frac{x_0}{x_1}}\right) = \frac{1}{2}\log\left(\frac{x_1+x_0}{x_1-x_0}\right)$$

$$\forall\ \left|\frac{x_0}{x_1}\right| < 1$$

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$\eta(x)_r := \sqrt{x_2^2 + x_3^2 + x_4^2}$

$\eta(x)_\theta := \arccos\left(\frac{x_4}{\sqrt{x_2^2+x_3^2+x_4^2}}\right)$

$\eta(x)_\varphi := \operatorname{arctg}\left(\frac{x_3}{x_2}\right)$

<u>Cl.:</u> $\eta \circ \eta^{-1} = \mathbb{1}_{\mathbb{R}\times(0,\alpha^{-1})\times(0,2\pi)^2}$ $\qquad \eta^{-1}\circ\eta = \mathbb{1}_V$

<u>Pf.:</u> Skipped.

$\Longrightarrow$ η is a bijection. Note η and η^{-1} are smooth.

Hence $\eta : V \to \mathbb{R}\times(0,\alpha^{-1})\times(0,2\pi)^2$ is a valid chart.

We want now h_η:

$(\partial_t \eta^{-1}{}_0)(t,r,\theta,\varphi) = \alpha^{-1}\sqrt{1-\alpha^2 r^2}\,ch(\alpha t)\alpha = \sqrt{1-\alpha^2r^2}\,ch(\alpha t)$

$(\partial_t \eta^{-1}{}_1)(\quad) = \alpha^{-1}\sqrt{1-\alpha^2r^2}\,sh(\alpha t)\alpha = \sqrt{1-\alpha^2r^2}\,sh(\alpha t)$

$(\partial_t \eta^{-1}{}_j)(\quad) = 0 \qquad \forall\ j\in\{2,3,4\}$

$(\partial_r \eta^{-1}{}_0)(\quad) = -\frac{r\alpha}{\sqrt{1-\alpha^2r^2}}\,sh(\alpha t)$

$(\partial_r \eta^{-1}{}_1)(\quad) = -\frac{r\alpha}{\sqrt{1-\alpha^2r^2}}\,ch(\alpha t)$

$(\partial_r \eta^{-1}{}_2)(\quad) = \sin(\theta)\cos(\varphi)$

$(\partial_r \eta^{-1}{}_3)(\quad) = \sin(\theta)\sin(\varphi)$

$(\partial_r \eta^{-1}{}_4)(\quad) = \cos(\theta)$

$(\partial_\theta \eta^{-1}{}_0)(\quad) = 0$	$(\partial_\varphi \eta^{-1}{}_0)(\quad) = 0$
$(\partial_\theta \eta^{-1}{}_1)(\quad) = 0$	$(\partial_\varphi \eta^{-1}{}_1)(\quad) = 0$
$(\partial_\theta \eta^{-1}{}_2)(\quad) = r\cos(\theta)\cos(\varphi)$	$(\partial_\varphi \eta^{-1}{}_2)(\quad) = -r\sin(\theta)\sin(\varphi)$
$(\partial_\theta \eta^{-1}{}_3)(\quad) = r\cos(\theta)\sin(\varphi)$	$(\partial_\varphi \eta^{-1}{}_3)(\quad) = r\sin(\theta)\cos(\varphi)$
$(\partial_\theta \eta^{-1}{}_4)(\quad) = -r\sin(\theta)$	$(\partial_\varphi \eta^{-1}{}_4)(\quad) = 0$

$(h^\eta_{tt}\circ\eta^{-1})(\quad) = \left(\sqrt{1-\alpha^2r^2}\,ch(\alpha t)\right)^2 - \left(\sqrt{1-\alpha^2r^2}\,sh(\alpha t)\right)^2 = 1-\alpha^2r^2.$

$(h^\eta_{rr}\circ\eta^{-1})(\quad) = -\frac{r^2\alpha^2}{1-\alpha^2r^2} - 1 = -\frac{\cancel{r^2\alpha^2} + 1 \cancel{-\alpha^2r^2}}{1-\alpha^2r^2} = -\frac{1}{1-\alpha^2r^2}.$

$$(h^{\eta}_{\theta\theta} \circ \eta^{-1})(\ ) = -r^2$$

$$(h^{\eta}_{\varphi\varphi} \circ \eta^{-1})(\ ) = -r^2 \sin(\theta)^2$$

<u>Cl.:</u> All other components of h^η are zero.
<u>Pf.:</u> Skipped.

$$\Rightarrow \boxed{(h^\eta \circ \eta^{-1})(t,r,\theta,\varphi) = \mathrm{diag}\left(1-\alpha^2 r^2, -\frac{1}{1-\alpha^2 r^2}, -r^2, -r^2 \sin(\theta)^2\right)}$$

This is just the spherically symmetric static metric.
($\Rightarrow$ The $r=\alpha$ singularity is just an artifact of η, not h!)
This shows (twice) that h is a solution of the Einstein field equations: φ is the Friedman sol-n w/ $k=0$ and $a(t) = \alpha^{-1} e^{\alpha t}$

iii) I. A distinguished observer for an expanding spacetime is a trajectory $\gamma : \mathbb{R} \to M$ (i.e. the worldline of an observer) s.t. γ obeys the geodesic eq-n (i.e. it is free falling) and s.t. $\dot\gamma$ (the observer's 4-velocity) is invariant under space symmetries.

<u>Cl.:</u> $\gamma^\varphi(\tau) := (\tau, Y_1, Y_2, Y_3)$ $\forall \tau \in \mathbb{R}$ $\exists Y \in \mathbb{R}^3$ solves the geodesic eq-n $\nabla_{\dot\gamma} \dot\gamma = 0$.

<u>Pf.:</u> First we must compute the Christoffel symbols in φ:

<u>Cl.:</u> If g^φ is diagonal then

$$\Gamma^\varphi{}_{ijk} = 0$$
$$\Gamma^\varphi{}_{ijj} = -\tfrac{1}{2}(g^\varphi{}_{ii})^{-1} \partial_i g^\varphi_{jj}$$
$$\Gamma^\varphi{}_{iji} = \partial_j \log\left(\sqrt{|g^\varphi{}_{ii}|}\right)$$
$$\Gamma^\varphi{}_{iii} = \partial_i \log\left(\sqrt{|g^\varphi{}_{ii}|}\right)$$
$$\forall\ i \neq j,\ i \neq k,\ j \neq k$$

<u>Pf.:</u> Skipped.

We need to calculate

$$\ddot{\gamma}^{\varphi}{}_{i} + \Gamma^{\varphi}{}_{ijk}\, \dot{\gamma}^{\varphi}_{j} \dot{\gamma}^{\varphi}_{k} \overset{?}{=} 0$$

$$\dot{\gamma}^{\varphi} = (1,0,0,0) \qquad \ddot{\gamma}^{\varphi} = 0$$

$$\Rightarrow \Gamma^{\varphi}{}_{ijk}\, \dot{\gamma}^{\varphi}_{j} \dot{\gamma}^{\varphi}_{k} = \Gamma^{\varphi}{}_{i00} = 0$$ verify from eq-ns above.

Cl.: $\dot{\gamma}$ is invariant under sp. symmetries.

Pf.: The point zero is invariant under rotations and Lorentz boosts.

If we wanted a picture of γ we'd like to embed it back into $\mathbb{R}^5$ via φ^{-1}. Then

$$\gamma^{\psi} = \psi \circ \varphi^{-1} \circ \gamma^{\varphi} \underset{\psi = \mathbb{1}}{=} \varphi^{-1} \circ \gamma^{\varphi}$$

If we wanted just the shape of γ^{ψ} (i.e. $\mathrm{im}(\gamma^{\psi})$) without the actual map we'd eliminate τ. We can do this via φ where we find:

$$\begin{cases} \dfrac{\gamma^{\psi}_{2}}{\gamma^{\psi}_{0} + \gamma^{\psi}_{1}} = Y_1 \\[2ex] \dfrac{\gamma^{\psi}_{3}}{\gamma^{\psi}_{0} + \gamma^{\psi}_{1}} = Y_2 \\[2ex] \dfrac{\gamma^{\psi}_{4}}{\gamma^{\psi}_{0} + \gamma^{\psi}_{1}} = Y_3 \end{cases}$$

So $\mathrm{im}(\gamma^{\psi})$ is the locus of points obeying these three eq-ns, intersection with $\mathcal{M}$.

II. A distinguished observer for the static spacetime is a trajectory $\lambda : \mathbb{R} \to \mathcal{M}$ s.t. $\dot{\lambda}$ is parallel to d^{η}_{t}. Note λ need not obey the geodesic eq-n necessarily. Hence λ^{η} should have its last three components fixed, so that $\dot{\lambda}^{\eta} \propto (1,0,0,0)$.

We obtain a plane for each choice $(R, \Theta, \varnothing) \in (0, \alpha^{-1}) \times (0, 2\pi)^2$

$$\lambda^4_2 = R \sin(\Theta) \cos(\phi)$$

$$\lambda^4_3 = R \sin(\Theta) \sin(\phi)$$

$$\lambda^4_4 = R \cos(\Theta)$$

$\boxed{Q2}$ On Hawking's Singularity Theorem

$(\mathcal{M}, g)$ p.R. mnfld of signature $(+,-,-,-)$.

$\Sigma \subseteq \mathcal{M}$ spacelike 3-surface w/o boundary with normal u s.t. $g(u,u)=1$, $g(u,X)=0 \ \forall \ X \in T_p\Sigma$.

Denote the side of Σ distinguished by u its future side.

Define a $(0,2)$ symmetric tensor K on Σ called extrinsic curvature by

$$K(X,Y) := -g(\nabla_X u, Y)$$

$\forall \ X, Y$ v/flds on Σ.

Thm: Σ cpt., $tr(K) \leq C < 0$. $Ricci(\xi, \xi) \geq 0 \ \forall$ timelike v/fld ξ on $\mathcal{M}$.

$\Longrightarrow \exists$ timelike geodesic starting from the past side of Σ ending in a singularity of $\mathcal{M}$. It reaches it w/ proper time $\leq 3/|C|$.

i) By $\boxed{HW8}\boxed{Q1}$ (iv), $Ricci(\xi,\xi) \geq 0$ is the strong energy cond. if g solves the E.F.E.

ii) Let Σ be a time slice of the Friedmann model, $\underbrace{u^A = (1,0,0,0)}_{\text{in chart A}}$, (which is cpt. if $k=+1$)

in chart A: Coordinates (x^1, x^2, x^3) with

$x^4 = \sqrt{R_0^2 - kr^2}$ $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$

Recall this is a chart for a time slice.

Then the full spacetime metric is $g^A = dt^2 - a(t)^2 g_0 \ \exists \ a$.

Cl.: $\dot{a}(t_0) > 0$ (expansion $\exists \ t_0$) $\Longrightarrow \exists$ singularity in the past of Σ.

Pf.: At $(t_0, 0,0,0)$, g is diagonal.

Indeed, by (6.3) then $g_0^A = \mathrm{diag}(1,1,1)$.

Thus with (u, e_1, e_2, e_3) the chart basis of A, we have

$$
\begin{aligned}
K^A{}_{ij} \equiv K(e_i, e_j) &= -g(\nabla_{e_i} u, e_j) \\
&= -g(\nabla_{e_i} e_0, e_j) \\
&= -g^A{}_{jj}\, e^j(\nabla_{e_i} e_0) \\
&= -\underbrace{g^A{}_{jj}}_{=-a^2}\, \underbrace{\Gamma^A{}_{j i 0}}_{\text{pp. }54\ \frac{\dot a}{a}\delta_{ij}} \\
&= +a\dot a\,\delta_{ij}
\end{aligned}
$$

$$\Rightarrow \operatorname{tr}(K) = \underbrace{(g^{-1})_{ij}}_{-\frac{1}{a^2}} K_{ji} = -3\frac{\dot a}{a} =: c < 0 \text{ by hypothesis.}$$

Hence by Hawkins' singularity theorem we find the result.

Note $\frac{3}{|c|} = \left|\frac{a}{\dot a}\right| = |H(t_0)^{-1}|$.