Forms

A $p$-form on $\mathcal{M}$ is an anti-symmetric tensor field of type $(0,p)$.

Cl.1. Any $p$-form is identically zero if $p>n=\dim(\mathcal{M})$.

R.f. Take any basis of $T^p_0\mathcal{M}$; it has $n$ elements; $e_1, \ldots, e_n$.
   So, if $\omega$ is such a $p$-form, it is determined by its action on this basis. But since there must be two slots with the same basis vector, we always get $0$.

Cl.2. Any $n$-form has only one component in a basis.

R.f. Let $\omega$ be an $n$-form, $\{e_i\}$ a basis,
   Its components are
   $\omega(e_{i_1}, \ldots, e_{i_n}) = \omega(e_{i_1}, \ldots, e_{i_n})$
   
   Note by A.S. all indices must be different, and up to sign we may always bring it to the canonical form
   $\omega(e_{i_1}, \ldots, e_{i_n})$.

Define anti-symmetrization on $(0,p)$ tensors as:

$$\left(\mathcal{A}(\tau)\right)(x_{i_1}, \ldots, x_{i_p}) = \frac{1}{p!} \sum_{\pi \in S_p} \text{sgn}(\pi) \tau(x_{\pi(i_1)}, \ldots, x_{\pi(i_p)})$$

Note $\mathcal{A}^2 = \mathcal{A}$.

Define exterior product of a $p$-form $\omega_1$ with a $q$-form $\omega_2$:

$$\omega_1 \wedge \omega_2 = \frac{(p+q)!}{p!q!} \mathcal{A} \left( \omega_1 \otimes \omega_2 \right)$$

Note: - Associativity
   - Not commutative:
     $$\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1$$

Define the exterior derivative as a map from $p$-forms to $(p+1)$-forms as:

$$ (d\omega)(x_{i_1}, \ldots, x_{i_{p+1}}) = \sum_{j=1}^{p+1} (-1)^{j-1} X_j^i \omega(\mathcal{O}(x_{i_1}, \ldots, x_{i_{j-1}}, x_{i_{j+1}}, \ldots, x_{i_{p+1}})) + \sum_{j=1}^{p+1} (-1)^{i-j} \omega([x_j, x_{i_j}], x_{i_1}, \ldots, x_{i_{j-1}}, x_{i_{j+1}}, \ldots, x_{i_{p+1}})$$
Note: Linear
- Compatible with $\mathbf{N}^2$: $d(x_1, x_2) = (\mathrm{d}x_1(x_2) + (-1)^p \mathrm{d}x_2(x_1)) \
- N$-closed: $\delta^2 = 0$
- Compatible with vector fields: $(\mathrm{d}f)(X) = X(f)$.
  (Hence the notation)
- Compatible with mfd morphisms: $p^* \delta = \delta^* p^*$
  for any $p: M \to N$ morphism of mfd.

Def.: A $p$-form is exact if it lies in $\im(\partial)$.
- A $p$-form is closed if it lies in $\ker(\partial)$.

Note: $\delta$ defines a local exact sequence:

$$0 \to \Omega^0(M) \xrightarrow{\partial} \Omega^1(M) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^p(M) \xrightarrow{\partial} 0$$

where $\Omega^p(M)$ is the space of $p$-forms on $M$, which is star-shaped.

To say this sequence is exact means that the image of every map is equal to the kernel of the next map: $\im(\partial_p) = \ker(\partial_{p+1})$.

Indeed this follows by: $\delta^2 = 0$ implies $= 0$.

Poincaré lemma gives $= 0$ (look it up).

Integration

Def.: An orientation on $M$ is an atlas of charts $\mathcal{A}$ in two charts $\Phi, \Psi$, the transition map has positive det. if $\det(M^{\Phi}_{\Psi}) > 0$.

Note: Not every mfd is orientable. E.g., Möbius band.

Note: An $n$-form $\omega$ transforms as:

$$\omega_{i_1, \ldots, i_n} = N_{i_1}^{i_1'} \cdots N_{i_n}^{i_n'} \omega_{i_1', \ldots, i_n'} = \det(N_{i_j}^{i_j'}) \omega_{i_1', \ldots, i_n'}$$
Let \( w \) be a \( n \)-form whose support is contained with \( U \) where \( \varphi: U \to \mathbb{R}^n \). Then

\[
\int_{M} w = \int_{\mathbb{R}^n} w^0_{i_1 \ldots i_n} \varphi^1
\]

Note this is independent of the chart used by the change of variable formula: Let \( f: \mathbb{R}^n \to \mathbb{R} \) be integrable with \( \text{supp} \, \varphi \subseteq V \subseteq \text{Open}(\mathbb{R}^n) \),

\[
\int_{V} f = \int_{\text{proj}(V)} f \circ \gamma \left| \det(\text{Jacobian}(\gamma)) \right|
\]

where \( \gamma: V \to \mathbb{R}^n \) is injective and differentiable.

Hence

\[
\int_{\mathbb{R}^n} w^0_{i_1 \ldots i_n} \varphi^1 = \int_{\mathbb{R}^n} \det(N^{\varphi}) w^0_{i_1 \ldots i_n} \varphi^1
\]

Change of variable formula with \( \gamma \):

\[
\int_{\mathbb{R}^n} w^0_{i_1 \ldots i_n} \varphi^1 = \int_{\mathbb{R}^n} \det(N^{\varphi}) w^0_{i_1 \ldots i_n} \varphi^1 \left| \det(\text{Jacobian}(\gamma)) \right|
\]

Positive orientation and \( N^{\varphi} = (N^{\varphi})^t \)

\[
\text{Def.} \quad \text{A partition of unity on } M \text{ is a collection of scalars } \{ \varphi_k \}_{k=1}^q \subseteq M \text{ s.t. } \varphi_k \geq 0, \sum_k \varphi_k = 1
\]

s.t. \( \forall \, p \in M, \exists \, M \subseteq \text{Open}(M) \), \( p \in U \) and only finitely many \( \varphi_k \)'s are non-zero on \( M \).

\text{Def.} For an arbitrary \( w \) of open support,

\[
\int_{M} w = \sum_k \int_{\mathbb{R}^n} \varphi_k w
\]
Def. A manifold with a boundary is locally homeomorphic to \( \{ x \in \mathbb{R}^n \mid x_i > 0 \} \cong \mathbb{R}_+ \times \mathbb{R}^{n-1} \).

\( \partial M := \{ p \in M \mid \exists \phi, \text{ any chart containing } p \} \) is a manifold of dimension \( n-1 \).

Note: \( \partial M \) is also a manifold and if \( M \) is orientable it induces an orientation on \( \partial M \).

**Stokes' Theorem**

Let \( M \) be an \( n \)-manifold with a boundary, \( \omega \) is any \((n-1)\)-form. Then

\[
\int_M \omega = \int_{\partial M} \omega
\]

Def. \( (i_x(\omega))(x_1, \ldots, x_{p+1}) := \omega(x, x_1, \ldots, x_{p+1}) \) for any \( p \)-form \( \omega \).

Hence \( i_x \) maps \( p \)-forms to \( p-1 \)-forms.

**Gauss' Theorem**

Let \( X \) be a vector field, \( \eta \) be a volume form (non-zero at all points) \( n \)-form. Then

\[
\int_M d(i_x(\eta)) = \int_{\partial M} i_x(\eta)
\]