Integration

Forms

A p-form on $\mathcal{M}$ is an anti-symmetric tensor field of type $(0,p)$.

Cl.: Any p-form is identically zero if $p > n \equiv \dim(\mathcal{M})$.

Pf.: Take any basis of $T_q\mathcal{M}$: it has n elements: $e_1, \dots, e_n$. So if Ω is such a p-form, it is determined by its action on this basis. But since there must be two slots with the same basis vector, we always get 0.

Cl.: Any n-form has only one component in a basis.

Pf.: Let Ω be an n-form, $\{e_i\}_i$ a basis. Its components are

$\Omega_{i_1, \dots, i_n} \equiv \Omega(e_{i_1}, \dots, e_{i_n})$

Note by A.S. all indices must be different, and up to sign we may always bring it to the canonical form $\Omega(e_1, \dots, e_n)$.

Define anti-symmetrization on $(0,p)$ tensors as:

$$(\mathcal{A}(T))(X_1, \dots, X_p) := \frac{1}{p!} \sum_{\pi \in S_p} (\mathrm{sgn}(\pi))\, T(X_{\pi(1)}, \dots, X_{\pi(p)})$$

Note $\mathcal{A}^2 = \mathcal{A}$.

Define exterior product of a p_1-form Ω_1 w/ a p_2-form Ω_2:

$$\Omega_1 \wedge \Omega_2 := \frac{(p_1+p_2)!}{p_1!\, p_2!} \mathcal{A}(\Omega_1 \otimes \Omega_2)$$

Note: - Associative
- Not commutative: $\Omega_1 \wedge \Omega_2 = (-1)^{p_1 p_2} \Omega_2 \Omega_1$

Define the exterior derivative as a map from p-forms to $(p+1)$-forms as:

$$(d\Omega)(X_1, \dots, X_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i-1} X_i(\Omega(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{p+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \Omega([X_i, X_j], X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_{p+1})$$

Note:
- Linear
- Compatible with $\wedge$: $d(\Omega_1 \wedge \Omega_2) = (d\Omega_1)\wedge\Omega_2 + (-1)^{p_1}\Omega_1 \wedge d\Omega_2$
- Nilpotent: $d^2 = 0$
- Compatible with vector fields: $(df)(X) = X(f)$. (Hence the notation)
- Compatible with mnfld morphisms:
$$\varphi^* d = d\varphi^*$$
for any $\varphi: \mathcal{M} \to \mathcal{N}$ morphism of mnflds.

Def.:
- A p-form is exact if it lies in $\mathrm{im}(d)$.
- A p-form is closed if it lies in $\ker(d)$.

Note d defines a local exact sequence:

$$0 \hookrightarrow \Omega^0(\mathcal{U}) \xrightarrow{d_1} \Omega^1(\mathcal{U}) \xrightarrow{d_2} \cdots \xrightarrow{d_n} \Omega^n(\mathcal{U}) \xrightarrow{d} 0$$

where $\Omega^p(\mathcal{U})$ is the space of p-forms on $\mathcal{U} \in \mathrm{Open}(\mathcal{M})$ which is star-shaped.
To say this sequence is exact means that the image of every map is equal to the kernel of the next map: $\mathrm{im}(d_p) = \ker(d_{p+1})$
Indeed this follows by: $d^2 = 0$ gives $\subseteq$.
Poincaré lemma gives $\supseteq$. (look it up).

Integration

Def.: An orientation on $\mathcal{M}$ is an atlas of charts s.t. $\forall$ two charts φ, ψ, the transition map has positive det.: $\det(M^{\psi\varphi}) > 0$.

Note: Not every mnfld is orientable. E.g. Möbius band.

Note: An n-form ω transforms as:

$$\omega^{\psi}_{1,\dots,n} = N^{\psi\varphi}_{1 i_1} \cdots N^{\psi\varphi}_{n i_n} \omega^{\varphi}_{i_1,\dots,i_n} \overset{\text{by anti-symmetry}}{=} \det(N^{\psi\varphi})\, \omega^{\varphi}_{1,\dots,n}$$

Let ω be an n-form whose support is contained with $\mathcal{U}$ where $\varphi: \mathcal{U} \to \mathbb{R}^n$. Then

$$\int_M \omega := \int_{\mathbb{R}^n} \omega^{\varphi}_{1,\dots,n} \circ \varphi^{-1}$$

Note this is indep. of the chart used by the change of variable formula: Let $f: \mathbb{R}^n \to \mathbb{R}$ be integrable with supp in $V \in \mathrm{Open}(\mathbb{R}^n)$.

$$\int_V f = \int_{\eta^{-1}(V)} f \circ \eta \, |\det(\mathrm{Jacobian}(\eta))|$$

where $\eta: V \to \mathbb{R}^n$ injective and differentiable.

Hence

$$\int_{\mathbb{R}^n} \omega^{\psi}_{1,\dots,n} \circ \psi^{-1} = \int_{\mathbb{R}^n} \det(N^{\psi\varphi})\, \omega^{\varphi}_{1,\dots,n} \circ \psi^{-1}$$

Change of var. formula with $\eta := \psi \circ \varphi^{-1}$

$$= \int_{\mathbb{R}^n} \det(N^{\psi\varphi})\, \omega^{\varphi}_{1,\dots,n} \circ \varphi^{-1} \underbrace{|\det(\mathrm{Jacobian}(\psi \circ \varphi^{-1}))|}_{= M^{\psi\varphi}}$$

positive orientation and $M^{\psi\varphi} = (N^{\psi\varphi})^{-1}$

$$= \int_{\mathbb{R}^n} \omega^{\varphi}_{1,\dots,n} \circ \varphi^{-1}$$

Def.: A partition of unity on $\mathcal{M}$ is a collection of scalars $\{h_k\}_k \subseteq \mathcal{M}$ s.t. $h_k \geq 0$, $\sum_k h_k = 1$ (the const function)

s.t. $\forall p \in \mathcal{M}$, $\exists\, \mathcal{U} \in \mathrm{Open}(\mathcal{M})$: $p \in \mathcal{U}$ and only finitely many h_k's are non-zero on $\mathcal{U}$.

Def.: For an arbitrary ω of cpt support,

$$\int_M \omega := \sum_k \int h_k \omega$$

Def.: A manifold with a boundary is locally homeomorphic to $\{x \in \mathbb{R}^n \mid x_1 \geq 0\} =: \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$

$\partial \mathcal{M} := \{p \in \mathcal{M} \mid \psi(p)_1 = 0$ where $\psi : \mathcal{U} \to \mathcal{M}$ is any chart containing $p\}$.

Note: $\partial \mathcal{M}$ is also a manifold of dim $n-1$ and if $\mathcal{M}$ is orientable it induces an orientation on $\partial \mathcal{M}$.

Stokes' Theorem

Let $\mathcal{M}$ be an n-manifold with a boundary, ω is any $(n-1)$-form. Then

$$\boxed{\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \omega}$$

Def.: $(i_X(\omega))(X_1, \ldots, X_{p-1}) := \omega(X, X_1, \ldots, X_{p-1})$

for any p-form ω.

Hence i_X maps p forms to $p-1$ forms.

Gauss' Theorem

Let X be a vector field, η be a volume form (non-zero at all points n-form). Then

$$\boxed{\int_{\mathcal{M}} d(i_X(\eta)) = \int_{\partial \mathcal{M}} i_X(\eta)}$$