Note: A distinction between topological manifold and differentiable manifold. For the latter, the transition maps between the charts must be differentiable.

The point about manifolds is that they "look" locally like \( \mathbb{R}^n \) for some \( n \) (same \( n \) everywhere you look).

**Def:** A top. manifold \( X \) is a Hausdorff (paracompct, 2nd countable) top. sp. st. \( \forall x \in X \exists U \subseteq \text{Nbd}_{X}(x) \) s.t. \( U \cong \mathbb{R}^n \) for some \( n \in \mathbb{N} \) (same \( n \) for all points). (Isomorphism in the category of Top.)

**Note:** If we require \( X \) connected, then:
- For \( n=0 \) all top. manifols are discrete sp. classified by cardinality.
- For \( n=1 \), non-empty, paracompct, \( \mathbb{R} \) or \( S^1 \), unconnected are disjoint unions.
- For \( n=2 \), non-empty, cpt. \( S^2, \mathbb{RP}^2, \mathbb{RP}^2 \cup \mathbb{RP}^2 \) or \( B(\mathbb{R}^2) \) (real proj. plane) or connected.
- \( n \geq 3 \) possible but hard
- \( n \geq 4 \) impossible

**Counterexamples**

- \( \ast \bullet \bullet \) because zoomed in, \( \ast \) does not look like \( I \) or \( - \).
Line with two origins:

\[ \mathbb{R} \times \mathbb{R} / (x,0) \sim (x,1) \quad \forall x \in \mathbb{R}\setminus\{0\} \]

with quotient topology.

Not Hausdorff bcs. any nbhd. of \((0,1)\) contains \((0,0)\) and vice versa.

- Differentiable manifolds
- \(\mathbb{T}^n\) diff. manifold (explicit example)
- Counterexample: top. manifold but not differentiable
- Vector bundles, \(T_p M\)
- Tensor products

**Def**: Let \(X\) be a top. manifold. For any two homeomorphisms \(\varphi_i: U_i \to \mathbb{R}^n\) \(\forall i \in \{1,2,3\}\), we have the transition map \(\varphi_2 \circ \varphi_1^{-1}: \varphi_1(U_{12}) \to \varphi_2(U_{23})\) where \(\varphi_i(U_{ij}) \in \text{Open}(\mathbb{R}^n)\).

**Def**: Let \(k \in \mathbb{N}_{\geq 0}\) be given. A \(C^k\)-diff. manifold is a top. manifold \(X\) s.t. \(\exists\) covering of \(X\) by homeomorphisms (the covering is called an atlas) s.t. any transition map of any two homeomorphisms in the atlas is \(k\) times differentiable.

If \(k = \infty\) we say **smooth manifold**.
If $X$ is a top. manifold of dimension $n \in \{1, 2, 3\}$ then $X$ has a unique (up to diffeomorphisms) differentiable structure. This is false in higher dimensions.

Example: The Cusp

The graph of $t \mapsto (t^3, t^{1/3})$ in $\mathbb{R}^2$

$X = \mathbb{R} / (t \sim \lambda t)$

Given $X$ the subspace topology of the set-top of $\mathbb{R}^2$.

Then (check) $X$ is a top. manifold, but it is not smooth bcs. of the cusp.

However, $\phi$ offers a homeomorphism of $\mathbb{R} \cong X$.

Since $\mathbb{R}$ has a smooth structure, we may pull it back to $X$ so that while not a smooth submanifold of $\mathbb{R}^2$, $X$ is still a smooth manifold.

Example: The 2-torus, $T^2$

Define $T^2 = \{ x \in \mathbb{R}^3 \mid (R-\sqrt{x_1^2+x_2^2})^2 + x_3^2 - r^2 = 0 \}$

for some $R > r > 0$.

Can verify this is a smooth submanifold of $\mathbb{R}^3$ as the zero set of the map $f_{R, r}$. 
can also show it is homeomorphic to 
\[ \mathcal{X} := (\mathbb{R} / 2\pi \mathbb{Z})^2 \approx S^1 \times S^1 \]

with quotient top. from subspace top. from \( \mathbb{R}^2 \).

Use that the product of manifolds is a manifold.

Or explicitly:

\[ \begin{array}{cc}
U_4 & U_1 \\
0 \leq \theta \leq \pi & 0 \leq \phi \leq 2\pi \\
U_3 & U_2 \\
-\pi \leq \theta \leq \pi & -\pi \leq \phi \leq \pi \\
\end{array} \]

Cover with four homeomorphisms and show transition maps are smooth. Let \( 0 < \varepsilon \ll 1 \) be given.

\[ U_1 := \{ x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi + \varepsilon), \]
\[ x_2 \in (-\varepsilon, \pi + \varepsilon) \} \]

\[ U_2 := \{ x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi + \varepsilon), \]
\[ x_2 \in (-\pi - \varepsilon, -\varepsilon) \} \]

\[ U_3 := \{ x \in \mathbb{R}^2 \mid x_1 \in (-\pi - \varepsilon, -\varepsilon), \]
\[ x_2 \in (-\varepsilon, \pi + \varepsilon) \} \]

\[ U_4 := \{ x \in \mathbb{R}^2 \mid x_1 \in (-\pi - \varepsilon, -\varepsilon), \]
\[ x_2 \in (-\pi - \varepsilon, -\varepsilon) \} \]

Define \( \varphi_i : U_i \rightarrow \mathbb{R}^2 \) as the inclusion map.

Consider the transition maps:

\[ U_1 \cap U_2 = \{ x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi + \varepsilon), \]
\[ x_2 \in (-\varepsilon, \varepsilon) \} \]

\[ (A_1) \quad \varphi_1^{-1} : \varphi_1(A_1) \rightarrow \mathcal{X}, \text{ given by the identity map.} \]

\[ (A_2) \quad \varphi_2 : A \rightarrow \varphi_2(A) \text{ given by the inclusion map.} \]

\[ \Rightarrow \text{The transition maps are all smooth.} \]
Let $V$ and $W$ be two $F$-vector spaces. Suppose we are given a bi-$F$-linear map $h : V \times W \to Z$. (h separately $F$-linear in each slot) $V \otimes W$ will be defined as a vector space s.t. $h$ can be factored through a linear map $\overline{h} : V \otimes W \to Z$.

\[
\begin{array}{c}
V \times W \overset{p}{\longrightarrow} V \otimes W \\
\downarrow h \quad \downarrow \overline{h} \\
Z
\end{array}
\]

To do that we must define a new $F$-vector space $V \otimes W$ and a bilinear map $p : V \times W \to V \otimes W$ s.t. $V$ bilinear $h$ is unique linear $f$ s.t. $[\overline{h} = h \circ p]$.

Define $V \otimes W$ as free $F$-spanned by the set $V \times W$, quotiented by the subspace

\[
N := \text{span} \{ x \in F(V \times W) \mid x = (v_1, w) + (v_2, w) - (v_1 + v_2, w) \\
x = (v, w_1) + (v, w_2) - (v, w_1 + w_2) \\
x = v(v, w) - (v, w) \\
x = v(v, w) - (v, dw)
\}
\]

for some $(v_1, v_2, v, w_1, w_2, w, x) \in V^3 \times W^3 \times F^3$.

Explicitly we write $[(v, w)] = \overline{v \otimes w}$.

Any vector of the form $v \otimes w$ is called simple.

A general vector is always a finite sum of simple vectors, $V \times W \otimes W$, $x = \sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W$.
Then the map $\Phi : V \times W \to V \otimes W$ is defined as:

$\Phi(v, w) = v \otimes w$

and by def, it is bi-$\mathcal{F}$-linear.

The factorization goes through as:

$h(x) := \sum_{i=1}^{n} h(v_i, w_i)$ where $x = \sum_{i=1}^{n} v_i \otimes w_i$.

Verify factorization:

$\Phi(v, w) \xrightarrow{\Phi} v \otimes w$

$\downarrow h$

$\downarrow h$

$h(v, w)$

$\checkmark$
Vector Bundles

$\mathcal{F}(M)$ is the algebra of smooth functions $M \to \mathbb{R}$ "near $p"$ at $p \in M$.

A tangent vector $X \in \mathcal{T}_p M$ is a linear map $\mathcal{F}(M) \to \mathbb{R}$, s.t. $X(fg) = X(g)p(f) + f(p) X(g) \quad \forall (f,g) \in \mathcal{F}(M)^2$.

The space of all such tangent vectors at $p \in M$ forms an $\mathbb{R}$-vector space $\mathcal{T}_p M$, called the tangent space to $M$ at $p$.

This defines a topological space $TM$, called the tangent bundle, defined as:

$$TM := \left\{ (p, X) \in M \times \bigcup_{p \in M} \mathcal{T}_p M \mid X \in \mathcal{T}_p M \right\}$$

$$= \bigsqcup_{p \in M} \mathcal{T}_p M \quad \text{(disjoint union topology)}$$

Also, if $M$ is contractible then $TM \cong M \times \mathbb{R}^m$.

This is how you should think of $TM$ "locally".

The tangent bundle is an example of a more general construction called a *vector bundle*.

Def.: Let $\mathcal{E}$ be a top. sp., called the base space. A family of top. sp. over $\mathcal{E}$ is the following data:

(a) $\mathcal{E}$ top. sp., called the total sp.
(b) $\pi : \mathcal{E} \to \mathcal{E}$, cont. and surjective, called the proj. map.
A family of \( \pi : E \to M \) is a **vector bundle** if it is *locally trivial*, that is, if for each \( p \in M \) there is an open neighborhood \( U \) of \( p \) such that the restricted family \( \pi : \pi^{-1}(U) \to U \) is the trivial product family, that is, \( \exists \) homeomorphism \( \varphi : U \times \mathbb{R}^n \to \pi^{-1}(U) \) s.t.
\[
\pi \circ \varphi = \text{proj}_U : U \times \mathbb{R}^n \to U
\]
for each \( U \), \( \varphi(U, \cdot) \) is a \( \mathbb{R}^n \)-isomorphism.

The point is that most vector space functors (dual, direct sum, tensor product, ...) become vector bundle functors.

\[
\Rightarrow \text{ Cotangent bundle is the dual bundle to the tangent bundle, for example.}
\]

A vector field is a cont. map \( \xi : M \to E \) which is compatible w/ \( \pi : E \to M \).