15:45 — 17:30   HIT F 31.1

GR — Recitation Session #1 ⑥   21/9/2017   1

⑥

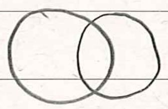
Note: $\exists$ distinction between topological manifold and differentiable manifold. For the latter the transition maps between the charts must be <u>differentiable</u>.

The point about manifolds is that they "look" locally like $\mathbb{R}^n$ for some n (same n everywhere you look).

Def: A top. manifold X is a Hausdorff (paracpt., 2^{nd} countable) top. sp. s.t. $\forall\ x \in X\ \exists\ U \in \mathrm{Nbhd}_X(x)$ s.t. $U \cong \mathbb{R}^n$ for some $n \in \mathbb{N}_{\geq 0}$ (same n for all points). (Isomorphism in the category of Top.)

Note: If we require X connected, then:
for $n=0$ all top. mnflds. are discrete sp. classified by cardinality,
for $n=1$, non-empty, paracpt. $\mathbb{R}$ or S^1.
unconnected are disjoint unions.
$n=2$, non-empty, cpt. S^2, $\mathbb{T}^2$, $\mathbb{T}^2 \vee \ldots \vee \mathbb{T}^2$ or $\mathbb{P}(\mathbb{R}^3)$ (real proj. plane) or connected sums.
$n=3$ possible but hard
$n \geq 4$ impossible

Counterexamples

(*)  because zoomed in, + does not look like | or —.

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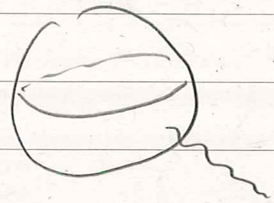
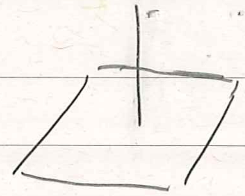
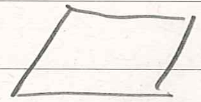
 because zoomed in 

does not look like  or .

(*) Line with two origins:

$\mathbb{R} \amalg \mathbb{R} \,/\, (x,A) \sim (x,B) \quad \forall\, x \in \mathbb{R}\setminus\{0\}$,
with quotient topology.

Not Hausdorff bcs. any nbhd. of $(0,A)$ contains $(0,B)$ and vice versa.

- ✓(*) Differentiable mnflds.
- ✓(*) $\mathbb{T}^2$ diff. mnfld. (explicit example).
- ✓(*) Counterexample: top. mnfld. but not differentiable.
- (*) Vector bundles. T_pM.
- ✓(*) Tensor products.

Def.: Let X be a top. mnfld. For any two homeomorphisms $\phi_i : U_i \xrightarrow{\sim} \mathbb{R}^n$ $\forall\, i \in \{1,2\}$, we have the transition map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ where $\phi_i(U_1 \cap U_2) \in \mathrm{Open}(\mathbb{R}^n)$.

Def.: Let $k \in \mathbb{N}_{\geq 1} \cup \{\infty\}$ be given. A C^k-diff.-mnfld is a top. mnfld X s.t. $\exists$ covering of X by homeomorphisms (the covering is called an atlas) s.t. any transition map of any two homeomorphisms in the atlas is k times differentiable.
If $k = \infty$ we say smooth manifold.

Cl.: If X is a top. mnfld. of dimension $n \in \{1,2,3\}$ then X has a unique (up to diffeomorphisms) differentiable structure. This is false in higher dimensions.

Cl.: $\exists$ 4-dim. cpt. simply-connected top. manifold which does not admit a smooth structure. (E_8)

Example: The Cusp (or graph of $|\cdot| : \mathbb{R} \to [0,\infty)$ in $\mathbb{R}^2$)

$X := im(\varphi)$ with $\varphi : \mathbb{R} \to \mathbb{R}^2$ given by

$\varphi(t) := t^2 e_1 + t^3 e_2$

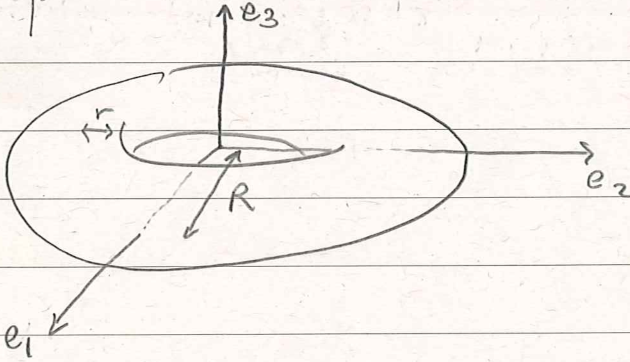
Give X the subsp. top. of the std. top. of $\mathbb{R}^2$. Then (check) X is a top. mnfld., but it is not smooth bcs. of the cusp.

However, φ offers a homeomorphism of $\mathbb{R} \cong X$. Since $\mathbb{R}$ has a smooth structure, we may pull it back to X so that while not a smooth submnfld of $\mathbb{R}^2$, X is still a smooth manifold.

Example: The 2-torus, $\mathbb{T}^2$

Define $\mathbb{T}^2 := \{x \in \mathbb{R}^3 \mid \underbrace{(R - \sqrt{x_1^2 + x_2^2})^2 + x_3^2 - r^2}_{=: f_{R,r}(x)} = 0\}$

for some $R > r > 0$.



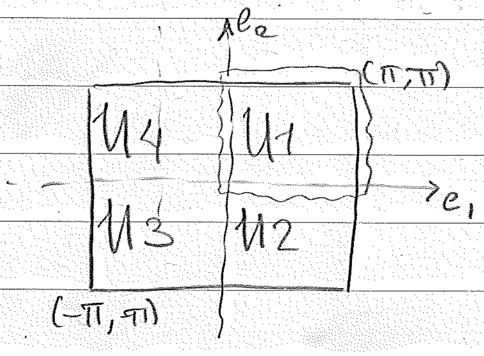
Can verify this is a smooth submnfld of $\mathbb{R}^3$ as the zero of the map $f_{R,r}$.

Can also show it is homeomorphic to

$$X := \underbrace{(\mathbb{R}/2\pi\mathbb{Z})^2}_{\text{with quotient top. from subsp. top. from std. top.}} \cong S^1 \times S^1$$

Use that the product of mnflds. is a mnfld.

Or explicitly:



Cover with four homeomorphisms and show transition maps are smooth: Let $0 < \varepsilon \ll 1$ be given.

$$U_1 := \{x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi+\varepsilon) \wedge x_2 \in (-\varepsilon, \pi+\varepsilon)\}$$

$$U_2 := \{x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi+\varepsilon) \wedge x_2 \in (-\pi-\varepsilon, \varepsilon)\}$$

$$U_3 := \{x \in \mathbb{R}^2 \mid x_1 \in (-\pi-\varepsilon, \varepsilon) \wedge x_2 \in (-\pi-\varepsilon, \varepsilon)\}$$

$$U_4 := \{x \in \mathbb{R}^2 \mid x_1 \in (-\pi-\varepsilon, \varepsilon) \wedge x_2 \in (-\varepsilon, \pi+\varepsilon)\}$$

Define $\phi_i : U_i \hookrightarrow \mathbb{R}^2$ as the inclusion map.

Consider the transition maps:

$$U_1 \cap U_2 = \underbrace{\{x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi+\varepsilon) \wedge x_2 \in (-\varepsilon, \varepsilon)\}}_{=: A} \cup \underbrace{\{x \in \mathbb{R}^2 \mid x_1 \in (-\varepsilon, \pi+\varepsilon) \wedge x_2 \in (\pi-\varepsilon, \pi+\varepsilon)\}}_{=: B}$$

$\phi_1^{-1} : \phi_1(A) \to X$ given by the identity map.
$\phi_2 : A \to \phi_2(A)$ given by the inclusion map.
$\Rightarrow$ The transition maps are all smooth.

Tensor Products

Let V and W be two $\mathbb{F}$-vector spaces.
Suppose we are given a bi-$\mathbb{F}$-linear map $h: V\times W \to Z$.
(h separately $\mathbb{F}$-linear in each slot)
$V\otimes W$ will be defined as a vector space s.t. h can be factored $\underbrace{\text{through}}_{\text{uniquely}}$ a <u>linear</u> map $\tilde{h}: V\otimes W \to Z$.

$$\begin{tikzcd} V\times W \arrow[r, "\varphi"] \arrow[dr, "h"'] & V\otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

To do that we must define a new $\mathbb{F}$-vector sp. $V\otimes W$ and a bilinear map $\varphi: V\times W \to V\otimes W$ s.t. $\forall$ bilinear h $\exists$ unique linear $\tilde{h}$ s.t. $\boxed{h = \tilde{h}\circ\varphi}$.

Define $V\otimes W$ as free v/sp. spanned by the set $V\times W$, quotiened by the subsp.

$$\begin{aligned} N := \operatorname{span}\{ x\in F(V\times W) \mid\ & x = (v_1,w)+(v_2,w)-(v_1+v_2,w) \\ & x = (v,w_1)+(v,w_2)-(v,w_1+w_2) \\ & x = \alpha(v,w)-(\alpha v,w) \\ & x = \alpha(v,w)-(v,\alpha w) \\ & \text{for some } (v_1,v_2,v,w_1,w_2,w,\alpha)\in V^3\times W^3\times\mathbb{F} \}) \end{aligned}$$

Explicitly we write $[(v,w)] =: v\otimes w$.
Any vector of the form $v\otimes w$ is called <u>simple</u>.
A general vector is always a finite sum of simple vectors: $\forall\ x\in V\otimes W$, $x=\sum_{i=1}^{n} v_i\otimes w_i$ $\exists$ $\{v_i\}_i \subseteq V$, $\{w_i\}_i \subseteq W$.

Then the map $\varphi : V \times W \to V \otimes W$ is def. as

$$(v,w) \mapsto v \otimes w$$

and by def. it is bi-$\mathbb{F}$-linear.

The factorization goes through as:

$\tilde{h}(x) := \sum_{i=1}^{n} h(v_i, w_i)$ where $x = \sum_{i=1}^{n} v_i \otimes w_i$.

Verify factorization:

$$\begin{tikzcd}
(v,w) \arrow[r, mapsto, "\varphi"] \arrow[dr, mapsto, "h"'] & v \otimes w \arrow[d, mapsto, "\tilde{h}"] \\
& h(v,w)
\end{tikzcd}$$

✓

Vector Bundles

$\mathcal{F}_p(\mathcal{M})$ is the algebra of smooth functions $\mathcal{M} \to \mathbb{R}$ "near p".

A tangent vector X at $p \in \mathcal{M}$ is a linear map $\mathcal{F}_p(\mathcal{M}) \to \mathbb{R}$, s.t. $X(fg) = X(f)\, g(p) + f(p)\, X(g) \quad \forall (f,g) \in \mathcal{F}_p(\mathcal{M})^2$.

The space of all such tangent vectors at $p \in \mathcal{M}$ forms an $\mathbb{R}$-v/sp. $T_p\mathcal{M}$ called the tangent space to $\mathcal{M}$ at p.

This defines a topological space $T\mathcal{M}$ called the tangent bundle, defined as:

$$T\mathcal{M} := \left\{ (p, X) \in \mathcal{M} \times \bigsqcup_{q \in \mathcal{M}} T_p\mathcal{M} \;\middle|\; X \in T_p\mathcal{M} \right\}$$

$$\equiv \bigsqcup_{p \in \mathcal{M}} T_p\mathcal{M} \qquad \text{(disjoint union topology)}$$

Cl.: If $\mathcal{M}$ is contractible then $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^m$.

This is how you should think of $T\mathcal{M}$ "locally".

The tangent bundle is an example of a more general construction called a vector bundle:

Def.: Let $\mathcal{M}$ be a top. sp., called the base space. A family of v/sp. over $\mathcal{M}$ is the following data:

(a) E top. sp., called the total sp.

(b) $\pi : E \to \mathcal{M}$ cont. and surjective, called the proj. map.

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(c) $\forall\, p \in \mathcal{M}$, $\pi^{-1}(\{p\})$ is an $\mathbb{R}$-v/sp., called the fiber over p, s.t. its v/sp. operations (addition and $\mathbb{R}$-mul.) are cont. w.r.t. subsp. top. on $\pi^{-1}(\{p\})$.

Def: A family of v/sp. over $\mathcal{M}$, $\pi : E \to \mathcal{M}$, is a vector bundle iff it is locally trivial, that is, iff $\forall\, p \in \mathcal{M}$ $\exists\, \mathcal{U} \in \mathrm{Nbhd}_{\mathcal{M}}(p)$ s.t. the restricted family $\pi : \pi^{-1}(\mathcal{U}) \to \mathcal{U}$ is the trivial product family, that is, $\exists$ homeomorphism $\varphi : \mathcal{U} \times \mathbb{R}^n \to \pi^{-1}(\mathcal{U})$ s.t.

$$\pi \circ \varphi = \mathrm{proj}_1 : \mathcal{U} \times \mathbb{R}^n \to \mathcal{U}$$

for each $u \in \mathcal{U}$, $\varphi(u, \cdot)$ is a v/sp. isomorphism.

The point is that most vector space functors (dual, direct sum, tensor product, ...) become vector bundle functors.

$\Rightarrow$ Cotangent bundle is the dual bundle to the tangent bundle, for example.

A vector field is a cont. map $v : \mathcal{M} \to E$ which is compatible w/ $\pi : E \to \mathcal{M}$.