GR–Recitation Session of Week 2 Summary

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Contents

1 The Basis of the Tangent Space induced by a Chart

Let $\varphi : U_\varphi \to \mathbb{R}^n$ and $\psi : U_\psi \to \mathbb{R}^n$ be two charts near some $p \in \mathcal{M}$.

Then we define basis vectors of $T_p\mathcal{M}$ corresponding to these charts as $d_i^\varphi := \left[\partial_i\left(\cdot \circ \varphi^{-1}\right)\right] \circ \varphi$. Note that this is really a vector *field* defined in a neighborhoud of p. In a point $q \in \mathcal{M}$ it is a tangent vector: d_i^φ at q is $\partial_i|_{\varphi(q)}\left(\cdot \circ \varphi^{-1}\right)$. There are analogous definitions for ψ. We define the expansion coefficients of a vector field X in the basis corresponding to φ as X_i^φ:

$$X = X_i^\varphi d_i^\varphi$$

so that $X_i^\varphi \equiv X(\varphi_i)$ with $\varphi_i := \pi_i \circ \varphi$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ is the natural projection. The transition rule (going from φ to ψ) for the expansion coefficients may be derived easily as

$$\begin{aligned} X_i^\psi &\equiv X(\psi_i) \\ &= X_j^\varphi d_j^\varphi(\psi_i) \end{aligned}$$

so that we define

$$M_{ij}^{\psi\varphi} := d_j^\varphi(\psi_i)$$

and get

$$X_i^\psi = M_{ij}^{\psi\varphi} X_j^\varphi$$

Similarly, we can move the basis vectors themselves:

$$\begin{aligned} d_i^\psi &= d_i^\psi(\varphi_j)\, d_j^\varphi \\ &= M_{ji}^{\varphi\psi} d_j^\varphi \\ &=: N_{ij}^{\psi\varphi} d_j^\varphi \end{aligned}$$

We also have a natural basis for $(T_p\mathcal{M})^*$, given by the dual of d_i^φ. Explicitly it is given by

$$e_i^\varphi := \cdot(\varphi_i)$$

That is, given any tangent vector X, $e_i^\varphi(X) \equiv X(\varphi_i) = X_i^\varphi$. The expansion coefficients of a 1-form ω are given by

$$\omega_i^\varphi = \omega\left(d_i^\varphi\right)$$

so that

$$\omega = \omega_i^\varphi e_i^\varphi$$

and the transformation rule for the expansion coefficients is

$$\begin{aligned}\omega_i^\psi &\equiv \omega\left(d_i^\psi\right)\\ &= \omega_j^\varphi e_j^\varphi\left(d_i^\psi\right)\end{aligned}$$

But $e_j^\varphi\left(d_i^\psi\right) \equiv d_i^\psi\left(\varphi_j\right) = N_{ij}^{\psi\varphi}$ so that we get

$$\omega_i^\psi = N_{ij}^{\psi\varphi}\omega_j^\varphi$$

and of course the dual basis vectors transform again in the opposite way compared to the expansion coefficients:

$$\begin{aligned}e_i^\psi &= e_i^\psi\left(d_j^\varphi\right)e_j^\varphi\\ &= d_j^\varphi\left(\psi_i\right)e_j^\varphi\\ &= M_{ij}^{\psi\varphi}e_j^\varphi\end{aligned}$$

We find that the expansion coefficients of a general $(k,\, l)$ tensor T transform as

$$T^\psi_{i_1\cdots i_k j_1\cdots j_l} = M^{\psi\varphi}_{i_1 i_1'}\cdots M^{\psi\varphi}_{i_k i_k'}N^{\psi\varphi}_{j_1 j_1'}\cdots N^{\psi\varphi}_{j_l j_l'}T^\varphi_{i_1'\cdots i_k' j_1'\cdots j_l'}$$

2 Properties of the Transition Matrices

1 Claim. We have $N_{ij}^{\psi\varphi}M_{ik}^{\psi\varphi} = \delta_{jk}$ and $N_{ij}^{\psi\varphi}M_{kj}^{\psi\varphi} = \delta_{ik}$.

Proof. We start by plugging in the definitions

$$N_{ij}^{\psi\varphi}M_{ik}^{\psi\varphi} \equiv d_i^\psi\left(\varphi_j\right)d_k^\varphi\left(\psi_i\right)$$

we swap out φ_j and ψ_i for e_j^φ and e_i^ψ respectively, because it is more transparent then that these are dual vectors to the d's. We get

$$\begin{aligned}N_{ij}^{\psi\varphi}M_{ik}^{\psi\varphi} &= d_i^\psi\left(e_j^\varphi\right)d_k^\varphi\left(e_i^\psi\right)\\ &= d_j^{\varphi*}\left(d_i^\psi\right)d_i^{\psi*}\left(d_k^\varphi\right)\\ &= \left\langle d_j^\varphi,\, d_i^\psi\right\rangle\left\langle d_i^\psi,\, d_k^\varphi\right\rangle\\ &= \left\langle d_j^\varphi,\, d_i^\psi\otimes d_i^{\psi*}d_k^\varphi\right\rangle\end{aligned}$$

Now we use the fact that $d_i^\psi\otimes d_i^{\psi*} = \mathbb{1}$ because $\left\{\, d_i^\psi \,\right\}_{i=1}^n$ is an ONB of $T_p\mathcal{M}$ for each p in the domain of that basis. Thus

$$N_{ij}^{\psi\varphi}M_{ik}^{\psi\varphi} = \left\langle d_j^\varphi,\, d_k^\varphi\right\rangle$$

and again using the fact that $\{\, d_i^\varphi \,\}_{i=1}^n$ is a basis one obtains the proper result. The other result is obtained by repeating the argument with $\varphi\leftrightarrow\psi$. □

2 Corollary. *We have* $d_l^\varphi\left(N_{ij}^{\psi\varphi}\right)M_{ik}^{\psi\varphi} = -N_{ij}^{\psi\varphi}d_l^\varphi\left(M_{ik}^{\psi\varphi}\right)$.

Proof. Apply d_l^φ on the foregoing equation. Since δ_{ik} is a constant scalar function, we get zero on the left hand side (as a tangent vector working on any scalar function is zero). On the right hand side we use the Leibniz property of d_l^φ. □

3 Claim. We have $d_k^\varphi\left(M_{ii'}^{\psi\varphi}\right) = d_{i'}^\varphi\left(M_{ik}^{\psi\varphi}\right)$.

Proof. If we expand out the definitions we will find that this boils down to the fact that $[d_i^\varphi,\, d_k^\varphi] = 0$, which is always true for basis tangent vectors which correspond to charts, which is what d_i^φ is. Indeed,

$$\begin{aligned}M_{ii',\,k} - M_{ik,\,i'} &\equiv d_k^\varphi\left(M_{ii'}\right) - d_{i'}^\varphi\left(M_{ik}\right)\\ &= d_k^\varphi\left(d_{i'}^\varphi\left(\psi_i\right)\right) - d_{i'}^\varphi\left(d_k^\varphi\left(\psi_i\right)\right)\\ &= [d_k^\varphi,\, d_{i'}^\varphi]\left(\psi_i\right)\end{aligned}$$

and $\left[d_i^{\varphi},\, d_j^{\varphi}\right]=0$ because

$$\begin{aligned}\left(\left[d_i^{\varphi},\, d_j^{\varphi}\right]\right)(f) &\equiv d_i^{\varphi} d_j^{\varphi} f-(i\leftrightarrow j)\\ &= \left[\partial_i\left(d_j^{\varphi} f\circ\varphi^{-1}\right)\right]\circ\varphi-(i\leftrightarrow j)\\ &= \left[\partial_i\left(\left[\partial_j\left(f\circ\varphi^{-1}\right)\right]\circ\varphi\circ\varphi^{-1}\right)\right]\circ\varphi-(i\leftrightarrow j)\\ &= \left[\partial_i\left(\partial_j\left(f\circ\varphi^{-1}\right)\right)\right]\circ\varphi-(i\leftrightarrow j)\\ &= 0\end{aligned}$$

as $\partial_i\partial_j=\partial_j\partial_i$. □

3 The Flow Generated by a Vector Field

Let $X\in\Gamma(T\mathcal{M})$ (that is, X is a section on the tangent bundle $T\mathcal{M}$, or equivalently, X is a vector field on $\mathcal{M}$).

Recall that a flow η is a group morphism $\eta:\mathbb{R}\to\operatorname{Aut}(\mathcal{M})$, where $\operatorname{Aut}(\mathcal{M})$ is the group of automorphisms of $\mathcal{M}$ considered as a smooth manifold, that is, the group of all diffeomorphisms $\mathcal{M}\to\mathcal{M}$. The flow η_X induced by a vector field X is the solution to the following differential equation

$$\begin{cases}\partial_t\left(\cdot\circ\left(\eta_X(t)\right)(p)\right) & = X\left(\left(\eta_X(t)\right)(p)\right) \qquad \forall p\in\mathcal{M}\\ \eta_X(0) & = \mathbb{1}_{\mathcal{M}}\end{cases}$$

Note what the first eqution actually means. Once we pick a scalar function $f\in\mathcal{F}_p(\mathcal{M})$, the map

$$t \quad\mapsto\quad f\circ\left(\eta_X(t)\right)(p)$$

is a function $\mathbb{R}\to\mathbb{R}$ so that the equation is a first order ordinary differential equation on that map.

By the Picard–Lindelöf theorem we know that there is a unique solution for such an ODE at least in a small enough neighborhoud. Thus in principle there is not a *global* flow $\eta_X:\mathbb{R}\to\operatorname{Aut}(\mathcal{M})$ associated to X but rather only a local flow $\eta_X^{\varepsilon}:(-\varepsilon,\,\varepsilon)\to\operatorname{Aut}(\mathcal{M})$ for $\varepsilon>0$ sufficiently small, near any give point $p\in\mathcal{M}$.

In class we looked at the following example which illustrates why the integration may not always be global:

4 Example. Let $\mathcal{M}:=\mathbb{R}$ and a vector field (given by its components) be given by $X(p):=p^2+1$. Then the flow $\gamma:\mathbb{R}\to\mathbb{R}$ must satisfy

$$\gamma' \quad=\quad \gamma^2+1$$

so that $\gamma=\tan(-C)$ for some constant $C\in\mathbb{R}$. To find it we employ the boundary condition: at time 0 the flow should land at p, so that $\gamma(0)\overset{!}{=}p$ and thus $C=-\arctan(p)$ and $\gamma_p(t):=\tan(t+\arctan(p))$. Clearly this is not everywhere define, for example, if $p=0$ the map is not define at $t\in\frac{\pi}{2}\mathbb{Z}$. Please make sure you understand how we got the description of a vector field as a derivation on scalar maps (which is how we think of it in the abstract setting of smooth manifolds) to this concrete example where it is merely a map $\mathbb{R}\to\mathbb{R}$. This goes through the identification in this example of $T_p\mathcal{M}\equiv T_p\mathbb{R}\cong\mathbb{R}$, so that a vector field $\mathcal{M}\to T_p\mathcal{M}$ is just a map $\mathbb{R}\to\mathbb{R}$.

4 The Pushforward and the Pullback

Let $\eta\in\operatorname{Aut}(\mathcal{M})$ be given. Then given any point $p\in\mathcal{M}$ we have defined a map

$$T_p\mathcal{M}\ni v \quad\mapsto\quad v(\cdot\circ\eta)\in T_{\eta(p)}\mathcal{M}$$

Indeed, If $f\in\mathcal{F}_p(\mathcal{M})$ then $f\circ\eta\in\mathcal{F}_{\eta(p)}(\mathcal{M})$ so that the tangent vector at $T_p\mathcal{M}$ "learns" how to act on scalar fields at $\eta(p)$ and is thus transformed into a tangent field at $T_{\eta(p)}\mathcal{M}$. So we have a map called *the differential*

$$\mathrm{d}\eta_p:T_p\mathcal{M} \quad\to\quad T_{\eta(p)}\mathcal{M}$$

given by

$$\mathrm{d}\eta_p(v) \quad:=\quad v(\cdot\circ\eta)$$

5 Claim. For any $p\in\mathcal{M}$, $\mathrm{d}\eta_p:T_p\mathcal{M}\to T_{\eta(p)}\mathcal{M}$ is a well-defined linear map.

If now $X\in\Gamma(T\mathcal{M})$ (that is, a vector *field* rather than a *tangent vector*) then since η is a diffeomorphism (in particular it is bijective) then we may also see how η applies on X rather than just on $X(p)\in T_p\mathcal{M}$ pointwise in $p\in\mathcal{M}$ (in fact η applied on X will define a *new* vector field). The definition goes as follows:

$$\eta_*:\Gamma(T\mathcal{M}) \quad\to\quad \Gamma(T\mathcal{M})$$

is called *the pushforward map* and is given by

$$\begin{aligned}(\eta_*(X))(p) &:= d\eta_{\eta^{-1}(p)}\left(X\left(\eta^{-1}(p)\right)\right) \quad \forall p\in\mathcal{M}\\ &= \mathcal{F}_p(\mathcal{M})\ni f\mapsto \left(X\left(\eta^{-1}(p)\right)\right)(f\circ\eta)\in\mathbb{R}\end{aligned}$$

Indeed, $X\left(\eta^{-1}(p)\right)\in T_{\eta^{-1}(p)}\mathcal{M}$ so that $d\eta_{\eta^{-1}(p)}:T_{\eta^{-1}(p)}\mathcal{M}\to T_p\mathcal{M}$ and so the definition makes sense and gives back a tangent vector in $T_p\mathcal{M}$, which is what we were trying to obtain.

Similarly, given a 1-form $\omega\in T^*\mathcal{M}$, for any point $p\in\mathcal{M}$ we get a dual tangent vector $\omega(p)\in T_p^*\mathcal{M}$ and so we may define a map

$$\eta^*:T^*\mathcal{M} \quad\to\quad T^*\mathcal{M}$$

via

$$((\eta^*(\omega))(p))(X) \quad := \quad (\omega(\eta(p)))(\mathrm{d}\eta_p(X)) \quad \forall X\in T_p\mathcal{M},\,\forall p\in\mathcal{M}$$

Indeed, $\mathrm{d}\eta_p(X)\in T_{\eta(p)}\mathcal{M}$ so that it makes sense for $\omega(\eta(p))\in T_p^*\mathcal{M}$ to act on it, and we get thus $(\eta^*(\omega))(p)\in T_p^*\mathcal{M}$.

In this sense we can define how to move general $(k,\,l)$ tensor field $T\in\underbrace{T\mathcal{M}\otimes\cdots\otimes T\mathcal{M}}_{k}\otimes\underbrace{T\mathcal{M}^*\otimes\cdots\otimes T\mathcal{M}^*}_{l}$:

$$\eta_*(T) \quad := \quad T\left(\eta^{*-1}\cdot,\,\cdots,\,\eta^{*-1}\cdot,\,\eta_*\cdot,\,\cdots,\,\eta_*\cdot\right)\circ\eta$$

5 The Lie Derivative

Let $X\in\Gamma(\mathcal{M})$ be given. The Lie derivative of a general $(k,\,l)$ tensor field $T\in\underbrace{T\mathcal{M}\otimes\cdots\otimes T\mathcal{M}}_{k}\otimes\underbrace{T\mathcal{M}^*\otimes\cdots\otimes T\mathcal{M}^*}_{l}$ is defined to be again a $(k,\,l)$ tensor field denoted by $\mathcal{L}_XT$ and given by the formula

$$\mathcal{L}_XT \quad := \quad \partial_t|_{t=0}\,\eta(t)^*(T)$$

where $\eta:(-\varepsilon,\,\varepsilon)\to\mathrm{Aut}(\mathcal{M})$ is the flow corresponding to X.

6 Claim. If $f\in\mathcal{F}(\mathcal{M})$ then $\mathcal{L}_Xf=X(f)$.

Proof. We have $\eta(t)^*(f)=f\circ\eta(t)$ so that $\partial_t|_{t=0}\,f\circ\eta(t)$ is just the definition of $X(f)$, since η is X's flow. □

7 Claim. If $Y\in\Gamma(\mathcal{M})$ then $\mathcal{L}_XY=[X,\,Y]$.

Proof. We have $\left(\eta(t)^*(Y)\right)(p)=(\eta(-t)_*(Y))(p)=d\eta(-t)_{\eta(-t)^{-1}(p)}\left(Y\left(\eta(-t)^{-1}(p)\right)\right)$ so that upon taking the derivative with time we should take into account the dependence on t coming from either $d\eta(-t)_{\eta(-t)^{-1}(p)}$ or $\eta(-t)^{-1}(p)$. Also note that because η is a group morphism $\eta(-t)^{-1}=\eta(t)$.

$$\begin{aligned}((\mathcal{L}_XY)(f))(p) &\equiv \partial_t|_{t=0}(\eta(-t)_*(Y))(f)\\ &= -\,\partial_t|_{t=0}\left(d\eta(t)_{\eta(t)^{-1}(p)}\left(Y\left(\eta(t)^{-1}(p)\right)\right)\right)(f)\\ &= -\,\partial_t|_{t=0}\,Y\left(\eta(t)^{-1}(p)\right)(f\circ\eta(t))\\ &= -\,\partial_t|_{t=0}\,Y(\eta(-t)(p))(f\circ\eta(t))\\ &\overset{*}{=} -Y(p)\left(\underbrace{\partial_t|_{t=0}\,f\circ\eta(t)}_{=X(f)}\right)-\underbrace{\underbrace{\partial_t|_{t=0}\,Y(\eta(-t)(p))(f)}\partial_t|_{t=0}\,Y(\cdot)(f)\circ\eta(-t)(p)}_{X(Y(f))}\\ &= -\,(Y(X(f))+X(Y(f)))(p)\\ &\equiv ([X,\,Y](p))(f)\end{aligned}$$

Let us elaborate one what happened at $*$: Essentially we have a map $\mathbb{R}^2\to\mathbb{R}$ given by

$$(s,\,t) \quad\mapsto\quad Y(\eta(s)(p))(f\circ\eta(t))$$

and we are trying to take the derivative of the map $\mathbb{R}\to\mathbb{R}$

$$t\mapsto(-t,\,t)\mapsto Y(\eta(-t)(p))(f\circ\eta(t))$$

Since this last map is a composition of two maps, the chain rule must be used. However here the derivative of the first map merely gives us either $+1$ or -1 and then we take the usual derivative varying only the first or second factor, so that we get:

$$\begin{aligned}\partial_t|_{t=0} Y(\eta(-t)(p))(f\circ\eta(t)) &= -\partial_s|_{s=0,\,t=0} Y(\eta(s)(p))(f\circ\eta(t)) + \partial_t|_{s=0,\,t=0} Y(\eta(s)(p))(f\circ\eta(t)) \\ & \quad (\eta(0)\equiv \mathbb{1}) \\ &= -\partial_s|_{s=0} Y(\eta(s)(p))(f) + \partial_t|_{t=0} Y(p)(f\circ\eta(t))\end{aligned}$$

For the second factor we have

$$\begin{aligned}\partial_t|_{t=0} Y(p)(f\circ\eta(t)) &\equiv \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(Y(p)(f\circ\eta(\varepsilon)) - Y(p)(f)\right) \\ & \quad (Y(p)\ \text{is linear and continuous}) \\ &= Y(p)\left(\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(f\circ\eta(\varepsilon) - f)\right) \\ &\equiv Y(p)\left(\partial_t|_{t=0} f\circ\eta(t)\right) \\ &\equiv Y(p)(X(f))\end{aligned}$$

For the first factor, we recognize that it is $X \equiv \partial_t|_{t=0}\cdot\circ\eta(t)$ working on the map $\mathcal{M}\ni q\mapsto Y(q)(f)\in\mathbb{R}$, which is an element of $\mathcal{F}(\mathcal{M})$ itself. ☐

8 Claim. If $\mu\in T^*\mathcal{M}$ then $\mathcal{L}_X\mu = X(\mu(\cdot)) - \mu([X,\,\cdot])$

Proof. Let $w\in T\mathcal{M}$. Then $\mu(w)\in\mathcal{F}(\mathcal{M})$. So

$$(\mathcal{L}_X\mu(w)) = X(\mu(w))$$

But one can also view $\mu(w)$ as contraction of the $(1,\,1)$ tensor $w\otimes\mu$: $\mu(w) =: \mathcal{C}(w\otimes\mu)$. It turns out that $\mathcal{L}_X$ commutes with contraction $\mathcal{C}$ (left as an exercise to the reader) so that

$$\begin{aligned}\mathcal{L}_X\mu(w) &= \mathcal{L}_X\mathcal{C}(w\otimes\mu) \\ &= \mathcal{C}\mathcal{L}_X w\otimes\mu\end{aligned}$$

It turns out (left as an exercise to the reader) that $\mathcal{L}_X$ obeys the Leibniz rule ($\mathcal{L}_X S\otimes T = (\mathcal{L}_X S)\otimes T + S\otimes\mathcal{L}_X T$). Thus we find

$$\begin{aligned}\mathcal{L}_X\mu(w) &= \mathcal{C}((\mathcal{L}_X w)\otimes\mu + w\otimes\mathcal{L}_X\mu) \\ &= \mathcal{C}([X,\,w]\otimes\mu + w\otimes\mathcal{L}_X\mu) \\ &\equiv \mu([X,\,w]) + (\mathcal{L}_X\mu)(w)\end{aligned}$$

We thus find the result since $w\in T\mathcal{M}$ was arbitrary. ☐

9 Corollary. *In this way we find an inductive formula for $\mathcal{L}_X$ working on a general $(k,\,l)$ tensor field:*

$$\begin{aligned}(\mathcal{L}_X T)(\mu_1,\,\cdots,\,\mu_k,\,v_1,\,\cdots,\,v_l) &= X(T(\mu_i,\,v_j)) - T(\mathcal{L}_X\mu_1,\,\mu_2,\,\cdots,\,\mu_k,\,v_j) - \cdots - T(\mu_1,\,\mu_2,\,\cdots,\,\mathcal{L}_X\mu_k,\,v_j) \\ & \quad -T(\mu_i,\,\mathcal{L}_X v_1,\,v_2,\,\cdots,\,v_l) - \cdots T(\mu_i,\,v_1,\,v_2,\,\cdots,\,\mathcal{L}_X v_l)\end{aligned}$$

for all $\{\,\mu_i\,\}_{i=1}^k\subseteq T^*\mathcal{M}$ and $\{\,v_j\,\}_{j=1}^l\subseteq T\mathcal{M}$.

Proof. Proceed as before inductively, again using the fact that $\mathcal{C}$ and $\mathcal{L}_X$ commute. See Wald appendix C for more details. ☐

10 Claim. The expansion coefficients of $\mathcal{L}_X T$ in the chart φ (using the notation as in the beginning of this document) are given by

$$\begin{aligned}(\mathcal{L}_X T)^\varphi_{i_1\cdots i_k j_1\cdots j_l} &= X^\varphi_r T^\varphi_{i_1\cdots i_k j_1\cdots j_l,\,r} - X^\varphi_{i_1,\,r}T^\varphi_{r\cdots i_k j_1\cdots j_l,\,r} - \cdots - X^\varphi_{i_k,\,r}T^\varphi_{i_1\cdots r j_1\cdots j_l,\,r} \\ & \quad + X^\varphi_{r,\,j_1}T^\varphi_{i_1\cdots i_k r\cdots j_l} + \cdots + X^\varphi_{r,\,j_l}T^\varphi_{i_1\cdots i_k j_1\cdots r}\end{aligned}$$

Proof. Use the above inductive formula together with the explicit expressions for $\mathcal{L}_X$ on scalars, vector fields and dual vector fields. Then use the definition of the expansion coefficients given in the beginning of the document. ☐