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1 The Basis of the Tangent Space induced by a Chart

Let \( \varphi : U_\varphi \to \mathbb{R}^n \) and \( \psi : U_\psi \to \mathbb{R}^n \) be two charts near some \( p \in M \).

Then we define basis vectors of \( T_pM \) corresponding to these charts as
\[
d_\varphi^i := \left[ \partial_i (\cdot \circ \varphi^{-1}) \right] \circ \varphi.
\]
Note that this is really a vector field defined in a neighborhood of \( p \). In a point \( q \in M \) it is a tangent vector:
\[
d_\varphi^i(q) = \partial_i |_{\varphi(q)} (\cdot \circ \varphi^{-1}).
\]
There are analogous definitions for \( \psi \). We define the expansion coefficients of a vector field \( X \) in the basis corresponding to \( \varphi \) as
\[
X^{\varphi}_i = X^{\varphi}(d_\varphi^i)
\]
so that \( X^{\varphi}_i \equiv X(\varphi_i) \) with \( \varphi_i := \pi_i \circ \varphi \) and \( \pi_i : \mathbb{R}^n \to \mathbb{R} \) is the natural projection. The transition rule (going from \( \varphi \) to \( \psi \)) for the expansion coefficients may be derived easily as
\[
X^{\psi}_i = X^{\psi}(d_\psi^i) = X^{\psi}(d_\psi^j(\psi_i)) = M^{\psi\varphi}_{ij} X^{\varphi}_j
\]
so that we define
\[
M^{\psi\varphi}_{ij} := d_\psi^j(\psi_i)
\]
and get
\[
X^{\psi}_i = M^{\psi\varphi}_{ij} X^{\varphi}_j
\]

Similarly, we can move the basis vectors themselves:
\[
d_\psi^i = d_\psi^i(\varphi_j) d_\varphi^j = M^{\psi\varphi}_{ij} d_\varphi^j = N^{\psi\varphi}_{ij} d_\psi^j
\]

We also have a natural basis for \( (T_pM)^* \), given by the dual of \( d_\varphi^i \). Explicitly it is given by
\[
e^{\varphi}_i := \cdot (\varphi_i)
\]
That is, given any tangent vector \( X \), \( e^{\varphi}_i(X) \equiv X(\varphi_i) = X^{\varphi}_i \). The expansion coefficients of a 1-form \( \omega \) are given by
\[
\omega^{\varphi}_i = \omega(d_\varphi^i)
\]
so that
\[
\omega = \omega^{\varphi}_i e^{\varphi}_i
\]
We have

Claim. We have

Properties of the Transition Matrices

Proof. We start by plugging in the definitions

we swap out \( \varphi_j \) and \( \psi_1 \) for \( e_j^\psi \) and \( e_i^\psi \) respectively, because it is more transparent then that these are dual vectors to the \( d_i^\psi \)’s. We get

Now we use the fact that \( d_i^\psi \otimes d_i^{\psi*} = 1 \) because \( \{ d_i^\psi \}_{i=1}^n \) is an ONB of \( T_p\mathcal{M} \) for each \( p \) in the domain of that basis. Thus

and again using the fact that \( \{ d_i^\psi \}_{i=1}^n \) is a basis one obtains the proper result. The other result is obtained by repeating the argument with \( \varphi \leftrightarrow \psi \).

Corollary. We have

Proof. Apply \( d_i^\psi \) on the foregoing equation. Since \( \delta_{ik} \) is a constant scalar function, we get zero on the left hand side (as a tangent vector working on any scalar function is zero). On the right hand side we use the Leibniz property of \( d_i^\psi \).

Claim. We have

Proof. If we expand out the definitions we will find that this boils down to the fact that \([d_i^\psi, d_k^\psi] = 0\), which is always true for basis tangent vectors which correspond to charts, which is what \( d_i^\psi \) is. Indeed,

\[
M_{ii', k} - M_{ik, i'} = d_k^\psi (M_{i'i}) - d_i^\psi (M_{ik}) = d_k^\psi (d_i^\psi (\psi_1)) - d_i^\psi (d_k^\psi (\psi_1)) = [d_k^\psi, d_i^\psi] (\psi_1)
\]
and \([d^\mu, d^\nu] = 0\) because
\[
([d^i, d^j]) (f) = d^i d^j f - (i \leftrightarrow j)
\]
\[
= [\partial_i (d^j f \circ \varphi^{-1})] \circ \varphi - (i \leftrightarrow j)
\]
\[
= [\partial_i (\partial_j (f \circ \varphi^{-1}) \circ \varphi \circ \varphi^{-1})] \circ \varphi - (i \leftrightarrow j)
\]
\[
= [\partial_i (\partial_j (f \circ \varphi^{-1})) \circ \varphi - (i \leftrightarrow j)
\]
\[
= 0
\]
as \(\partial_i \partial_j = \partial_j \partial_i\).

\[\square\]

3 The Flow Generated by a Vector Field

Let \(X \in \Gamma (T \mathcal{M})\) (that is, \(X\) is a section on the tangent bundle \(T \mathcal{M}\), or equivalently, \(X\) is a vector field on \(\mathcal{M}\)).

Recall that a flow \(\eta\) is a group morphism \(\eta : \mathbb{R} \rightarrow \text{Aut} (\mathcal{M})\), where \(\text{Aut} (\mathcal{M})\) is the group of all diffeomorphisms \(\mathcal{M} \rightarrow \mathcal{M}\) considered as a smooth manifold, that is, the group of all diffeomorphisms \(\mathcal{M} \rightarrow \mathcal{M}\). The flow \(\eta_X\) induced by a vector field \(X\) is the solution to the following differential equation

\[
\begin{cases}
\partial_t (\cdot \circ (\eta_X (t)) (p)) = X ((\eta_X (t)) (p)) & \forall p \in \mathcal{M} \\
\eta_X (0) = \mathbb{I} \mathcal{M}
\end{cases}
\]

Note what the first equation actually means. Once we pick a scalar function \(f \in \mathcal{F}_p (\mathcal{M})\), the map

\[t \mapsto f \circ (\eta_X (t)) (p)\]

is a function \(\mathbb{R} \rightarrow \mathbb{R}\) so that the equation is a first order ordinary differential equation on that map.

By the Picard–Lindelöf theorem we know that there is a unique solution for such an ODE at least in a small enough neighborhood. Thus in principle there is not a \textit{global} flow \(\eta_X : \mathbb{R} \rightarrow \text{Aut} (\mathcal{M})\) associated to \(X\) but rather only a \textit{local} flow \(\eta^*_X : (-\varepsilon, \varepsilon) \rightarrow \text{Aut} (\mathcal{M})\) for \(\varepsilon > 0\) sufficiently small, near any given point \(p \in \mathcal{M}\).

In class we looked at the following example which illustrates why the integration may not always be global:

4 Example. Let \(\mathcal{M} := \mathbb{R}\) and a vector field (given by its components) be given by \(X (p) := p^2 + 1\). Then the flow \(\gamma : \mathbb{R} \rightarrow \mathbb{R}\) must satisfy

\[\gamma' = \gamma^2 + 1\]

so that \(\gamma = \tan (-C)\) for some constant \(C \in \mathbb{R}\). To find it we employ the boundary condition: at time 0 the flow should land at \(p\), so that \(\gamma (0) = p\) and thus \(C = -\arctan (p)\) and \(\gamma_p (t) := \tan (t + \arctan (p))\). Clearly this is not everywhere defined, for example, if \(p = 0\) the map is not defined at \(t \in \frac{\pi}{2} \mathbb{Z}\). Please make sure you understand how we got the description of a vector field as a derivation on scalar maps (which is how we think of it in the abstract setting of smooth manifolds) to this concrete example where it is merely a map \(\mathbb{R} \rightarrow \mathbb{R}\). This goes through the identification in this example of \(T_p \mathcal{M} \equiv T_p \mathbb{R} \cong \mathbb{R}\), so that a vector field \(\mathcal{M} \rightarrow T_p \mathcal{M}\) is just a map \(\mathbb{R} \rightarrow \mathbb{R}\).

4 The Pushforward and the Pullback

Let \(\eta \in \text{Aut} (\mathcal{M})\) be given. Then given any point \(p \in \mathcal{M}\) we have defined a map

\[T_p \mathcal{M} \ni v \mapsto v (\cdot \circ \eta) \in T_{\eta(p)} \mathcal{M}\]

Indeed, If \(f \in \mathcal{F}_p (\mathcal{M})\) then \(f \circ \eta \in \mathcal{F}_{\eta(p)} (\mathcal{M})\) so that the tangent vector at \(T_p \mathcal{M}\) “learns” how to act on scalar fields at \(\eta(p)\) and is thus transformed into a tangent field at \(T_{\eta(p)} \mathcal{M}\). So we have a map called the \textit{differential}

\[d \eta : T_p \mathcal{M} \rightarrow T_{\eta(p)} \mathcal{M}\]

given by

\[d \eta_p (v) := v (\cdot \circ \eta)\]

5 Claim. For any \(p \in \mathcal{M}\), \(d \eta : T_p \mathcal{M} \rightarrow T_{\eta(p)} \mathcal{M}\) is a well-defined linear map.

If now \(X \in \Gamma (T \mathcal{M})\) (that is, a vector field rather than a tangent vector) then since \(\eta\) is a diffeomorphism (in particular it is bijective) then we may also see how \(\eta\) applies on \(X\) rather than just on \(X (p) \in T_p \mathcal{M}\) pointwise in \(p \in \mathcal{M}\) (in fact \(\eta\) applied on \(X\) will define a \textit{new} vector field). The definition goes as follows:

\[\eta_* : \Gamma (T \mathcal{M}) \rightarrow \Gamma (T \mathcal{M})\]
is called the pushforward map and is given by
\[
(\eta_*(X))(p) := d\eta_{\eta^{-1}(p)} (X (\eta^{-1}(p))) \quad \forall p \in M
\]

Indeed, \(X (\eta^{-1}(p)) \in T_{\eta^{-1}(p)}M\) so that \(d\eta_{\eta^{-1}(p)} : T_{\eta^{-1}(p)}M \rightarrow T_pM\) and so the definition makes sense and gives back a tangent vector in \(T_pM\), which is what we were trying to obtain.

Similarly, given a 1-form \(\omega \in T^*M\), for any point \(p \in M\) we get a dual tangent vector \(\omega (p) \in T^*_pM\) and so we may define a map

\[
\eta^* : T^*M \rightarrow T^*M
\]

via
\[
((\eta^*(\omega))(p))(X) := (\omega(\eta(p)))(d\eta_p(X)) \quad \forall X \in T_pM, \forall p \in M
\]

Indeed, \(d\eta_p(X) \in T_{\eta(p)}M\) so that it makes sense for \(\omega(\eta(p)) \in T^*_pM\) to act on it, and we get thus \((\eta^*(\omega))(p) \in T^*_pM\).

In this sense we can define how to move general \((k, l)\) tensor field \(T \in \underbrace{T_M \otimes \cdots \otimes T_M}_{p} \otimes \underbrace{T^*_M \otimes \cdots \otimes T^*_M}_{l}\):

\[
\eta_*(T) := T(\eta^{*-1}, \ldots, \eta^{*-1}, \eta_*^*, \ldots, \eta_*^*) \circ \eta
\]

5 The Lie Derivative

Let \(X \in \Gamma(M)\) be given. The Lie derivative of a general \((k, l)\) tensor field \(T \in \underbrace{T_M \otimes \cdots \otimes T_M}_{p} \otimes \underbrace{T^*_M \otimes \cdots \otimes T^*_M}_{l}\) is defined to be again a \((k, l)\) tensor field denoted by \(\mathcal{L}_X T\) and given by the formula

\[
\mathcal{L}_X T := \left. \partial_t \right|_{t=0} \eta(t)^*(T)
\]

where \(\eta : (-\varepsilon, \varepsilon) \rightarrow \text{Aut}(M)\) is the flow corresponding to \(X\).

6 Claim. If \(f \in \mathcal{F}(M)\) then \(\mathcal{L}_X f = X(f)\).

**Proof.** We have \(\eta(t)^*(f) = f \circ \eta(t)\) so that \(\left. \partial_t \right|_{t=0} f \circ \eta(t)\) is just the definition of \(X(f)\), since \(\eta\) is \(X\)’s flow.

7 Claim. If \(Y \in \Gamma(M)\) then \(\mathcal{L}_X Y = [X, Y]\).

**Proof.** We have \((\eta(t)^* (Y))(p) = (\eta(-t)^* (Y))(p) = d\eta(-t)_{\eta(-t)^{-1}(p)} (Y (\eta(-t)^{-1}(p)))\) so that upon taking the derivative with time we should take into account the dependence on \(t\) coming from either \(d\eta(-t)_{\eta(-t)^{-1}(p)}\) or \(\eta(-t)^{-1}(p)\). Also note that because \(\eta\) is a group morphism \(\eta(-t)^{-1} = \eta(t)\).

\[
((\mathcal{L}_X Y)(f))(p) = \left. \partial_t \right|_{t=0} (\eta(-t)^* (Y))(f) = - \left. \partial_t \right|_{t=0} \left( d\eta(t)_{\eta(t)^{-1}(p)} (Y (\eta(t)^{-1}(p))) \right)(f) = - \left. \partial_t \right|_{t=0} Y (\eta(t)^{-1}(p))(f \circ \eta(t)) = - \left. \partial_t \right|_{t=0} Y (\eta(-t)(p))(f \circ \eta(t)) \overset{*}{=} -Y(p) \left. \partial_t \right|_{t=0} f \circ \eta(t) = -X(f) = -Y(X(f)) = X(Y(f))(p)
\]

Let us elaborate one what happened at \(\ast\): Essentially we have a map \(\mathbb{R}^2 \rightarrow \mathbb{R}\) given by

\[
(s, t) \mapsto Y(\eta(s)(p))(f \circ \eta(t))
\]

and we are trying to take the derivative of the map \(\mathbb{R} \rightarrow \mathbb{R}\)

\[
t \mapsto (-t, t) \mapsto Y(\eta(-t)(p))(f \circ \eta(t))
\]

Since this last map is a composition of two maps, the chain rule must be used. However here the derivative of the first map...
merely gives us either $+1$ or $-1$ and then we take the usual derivative varying only the first or second factor, so that we get:

$$
\partial_{t=0} Y (\eta (-t) (p)) (f \circ \eta (t)) = - \partial_{s=0,t=0} Y (\eta (s) (p)) (f \circ \eta (t)) + \partial_{s=0,t=0} Y (\eta (s) (p)) (f \circ \eta (t))
$$

For the second factor we have

$$
\partial_{t=0} Y (p) (f \circ \eta (t)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (Y (p) (f \circ \eta (\varepsilon)) - Y (p) (f))
$$

For the first factor, we recognize that it is $X \equiv \partial_{t=0} \circ \eta (t)$ working on the map $\mathcal{M} \ni q \mapsto Y (q) (f) \in \mathbb{R}$, which is an element of $\mathcal{F} (\mathcal{M})$ itself.

\[8\text{ Claim.}\] If $\mu \in T^* \mathcal{M}$ then $\mathcal{L}_X \mu = X (\mu (\cdot)) - \mu ([X, \cdot])$

\textbf{Proof.} Let $w \in TM$. Then $\mu (w) \in \mathcal{F} (\mathcal{M})$. So

$$(\mathcal{L}_X \mu (w)) = X (\mu (w))$$

But one can also view $\mu (w)$ as contraction of the $(1, 1)$ tensor $w \otimes \mu$: $\mu (w) =: \mathcal{C} (w \otimes \mu)$. It turns out that $\mathcal{L}_X$ commutes with contraction $\mathcal{C}$ (left as an exercise to the reader) so that

$$\mathcal{L}_X \mu (w) = \mathcal{L}_X \mathcal{C} (w \otimes \mu) = \mathcal{C} \mathcal{L}_X \, w \otimes \mu$$

It turns out (left as an exercise to the reader) that $\mathcal{L}_X$ obeys the Leibniz rule ($\mathcal{L}_X S \otimes T = (\mathcal{L}_X S) \otimes T + S \otimes \mathcal{L}_X T$). Thus we find

$$\mathcal{L}_X \mu (w) = \mathcal{C} ([\mathcal{L}_X w] \otimes \mu + w \otimes \mathcal{L}_X \mu) = \mathcal{C} ([X, w] \otimes \mu + w \otimes \mathcal{L}_X \mu) = \mu ([X, w]) + (\mathcal{L}_X \mu) (w)$$

We thus find the result since $w \in TM$ was arbitrary.

\[9\text{ Corollary.}\] In this way we find an inductive formula for $\mathcal{L}_X$ working on a general $(k, l)$ tensor field:

$$(\mathcal{L}_X T) (\mu_1, \cdots, \mu_k, v_1, \cdots, v_l) = X (T (\mu_1, v_1)) - T (\mathcal{L}_X \mu_1, \mu_2, \cdots, \mu_k, v_j) - \cdots - T (\mu_1, \mu_2, \cdots, \mathcal{L}_X \mu_k, v_j)
$$

$$- T (\mu_1, \mathcal{L}_X v_1, v_2, \cdots, v_l) - \cdots - T (\mu_1, v_1, \mu_2, \cdots, \mathcal{L}_X v_l)$$

for all $\{ \mu_i \}_{j=1}^k \subseteq T^* \mathcal{M}$ and $\{ v_j \}_{j=1}^l \subseteq TM$.

\textbf{Proof.} Proceed as before inductively, again using the fact that $\mathcal{C}$ and $\mathcal{L}_X$ commute. See Wald appendix C for more details.

\[10\text{ Claim.}\] The expansion coefficients of $\mathcal{L}_X T$ in the chart $\varphi$ (using the notation as in the beginning of this document) are given by

$$
(\mathcal{L}_X T)_{i_1 \cdots i_k j_1 \cdots j_l} = \sum_{r=1}^l X_{r, j_1 \cdots j_l} T^\varphi_{i_1 \cdots i_k j_1 \cdots j_l, r} + \cdots + X_{r, j_l} T^\varphi_{i_1 \cdots i_k j_1 \cdots j_l, r}
$$

\textbf{Proof.} Use the above inductive formula together with the explicit expressions for $\mathcal{L}_X$ on scalars, vector fields and dual vector fields. Then use the definition of the expansion coefficients given in the beginning of the document.