General Relativity – Recitation Session of Week III — Parallel Transport

Let (as always) $\mathcal{M}$ be a smooth manifold of dimension n. So far in the course we have defined many structures on $\mathcal{M}$, some entailed a choice, others were entirely natural ($\equiv \natural$):

$$\text{Top. Manifold} \xrightarrow[\text{when } n \geq 4]{\text{choice}} \text{Smooth Structure}$$

$\natural$ (no choices made)

Tangent Bundle
Cotangent Bundle
Exterior Bundle
Tensor-Algebra Bundle

Parallel transport is an additional structure on $\mathcal{M}$ which entails a choice, and is hence not $\natural$.

Note that there are two more structures on $\mathcal{M}$ called covariant derivative and affine connection. They are both also not $\natural$. However it turns out that choosing one of these 3 determines the other two:

$$\underset{\text{Local}}{\boxed{\text{Parallel Transport}}} \longleftrightarrow \underset{\text{Infinitesimal}}{\boxed{\text{Covariant Derivative}}} \longleftrightarrow \boxed{\text{Affine Connection}}$$

Thus it is a matter of convention which structure to initially choose or specify so that the other two structures are determined.

Note: Any curve γ defines a family of tangent vectors along its image. We interpret the parallel transport as a way to move tangent vectors along the image of γ in a way that is "parallel" to $\dot{\gamma}$ (the tangent vectors induced by γ). Just what "parallel" means is specified by the choice of τ (no $\angle$ notion).

Notions of "parallel" coming from $\mathbb{R}^n$:

- Inner product:
$$\langle u, v\rangle_{\mathbb{R}^n} = \|u\|\,\|v\|\cos(\theta_{u,v})$$
angle between u, v

So if $\langle u, v\rangle_{\mathbb{R}^n} = \|u\|\,\|v\|$, $u \parallel v$.

- Colinearity:
$$u = \alpha v \quad \exists\ \alpha > 0$$

These notions may not be used on M via charts because they would end up being chart dependent, which makes them useless.

Given a parallel transport τ and a chart $\psi : U \to \mathbb{R}^n$, $\exists$ field Γ_ψ called "the Christoffel symbols" s.t.

$$-\partial_t\big|_{t=s} \left(\tau(\gamma)_{ts}\right)\left(d_i^\psi(\gamma(t)), e_j^\psi(\gamma(s))\right) = \Gamma_\psi(\gamma(s))_{i,jk}\, (\dot{\overline{\psi_k \circ \gamma}})(s)$$

where $\{e_i^\psi\}_i$, $\{d_i^\psi\}_i$ are the bases of T_p^*M, T_pM resp.

(see previous summary for the definitions)

Note $(\dot{\overline{\psi_k \circ \gamma}})$ is the k^{th} expan. coeff in $\{d_i^\psi\}_i$ of the vector field tangent to γ.

a) $TM \equiv \coprod_{p \in M} T_pM$ defines the tangent bundle.

Disjoint Union

If $\{X_i\}_{i \in I}$ is a family of top. sp. $D := \coprod_{i \in I} X_i$ is defined as follows: As a set,

$$D \equiv \left\{ (i, x) \in I \times \bigcup_{j \in I} X_j \;\middle|\; x \in X_i \right\}$$

We have canonical injections $\eta_i : X_i \hookrightarrow D$ $\quad \eta_i(x) := (i, x)$.

Open(D) is defined as the ~~finest~~ largest topology on D s.t. η_i is cont. $\forall i \in I$.

$\iff$ $\mathcal{U} \in \mathrm{Open}(D) \iff \eta_i^{-1}(\mathcal{U}) \in \mathrm{Open}(X_i) \quad \forall i \in I$.

For us the injections are $\eta_p : T_pM \hookrightarrow TM \quad \forall p \in M$.
$\qquad v \mapsto (p, v)$

Note we also have the ↯ proj. $\pi : TM \to M \quad (p, v) \mapsto p$.

$T_pM \equiv \pi^{-1}(\{p\}) \equiv (TM)_p$ is called the fiber over p.

Note $T_pM \cong \mathbb{R}^n$ as v/sp., but also as a topological v/sp. via the structure of M as a top. manifold.
$\iff$ The v/sp. operations at each p are continuous.

Also note that the subsp. top. of $(TM)_p$ is the same as the top. of T_pM previously described, by def. of the quotient top.

Thus TM is endowed with the structure of "a family of vector spaces over M." (See def. in Atiyah's K-Theory).

Since M is a top. manifold, $\forall p \in M \; \exists \; \psi : \mathcal{U} \to \mathbb{R}^n$ s.t.
$\mathcal{U} \in \mathrm{Open}(M)$ and ψ is a homeomorphism.

$\mathcal{M}$ n-dim mnfld points p

$T\mathcal{M}$ $2n$-dim mnfld points (p,v) $v \in T_p\mathcal{M}$

$T(T\mathcal{M})$ $4n$-dim mnfld points $((p,v),u)$ $u \in T_{(p,v)}(T\mathcal{M})$

$T_{(p,v)}(T\mathcal{M})$ $2n$-dim vecsp.

$\psi : \mathcal{U} \to \mathbb{R}^n$ chart on $\mathcal{M}$.

$\pi_{T\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$ proj : $\pi((p,v)) := p$

$$\hat{\psi} : \pi^{-1}(\mathcal{U}) \to \mathbb{R}^{2n}$$
$$(p,v) \mapsto \left(\psi(p), \begin{bmatrix} v(\psi_1) \\ \vdots \\ v(\psi_n) \end{bmatrix}\right)$$

$\pi_{T\mathcal{M}}$ has a push forward:

$(\pi_{T\mathcal{M}})_* : TT\mathcal{M} \to T\mathcal{M}$

It is given by: Let $V \in TT\mathcal{M}$.

Then $V = ((p,v),u)$ $\exists$ $p \in \mathcal{M}$, $v \in T_p\mathcal{M}$, $u \in T_{(p,v)}(T\mathcal{M})$.

$(\pi_{T\mathcal{M}})_*$ then maps $T_{(p,v)}(T\mathcal{M}) \to T_p\mathcal{M}$ by

$$u \longmapsto u(\cdot \circ \pi)$$

$\Rightarrow$ u only differentiates w.r.t. the change in base point, not w.r.t. change in tangent vector.

$(\pi_{T\mathcal{M}})_* : TT\mathcal{M} \to T\mathcal{M}$ makes $TT\mathcal{M}$ into a vector bundle too. Its fibre has the structure:

$(\pi_{T\mathcal{M}})_*^{-1}(\{(p,v)\}) \cong \{(p,v)\} \times \mathbb{R}^{2n}$

In a chart, the basis for $T_{(p,v)}(T\mathcal{M})$ is given by $\{d_i^{\hat{\psi}}\}_{i=1}^{2n}$

Let $f \in \mathcal{F}(T\mathcal{M})$. Then $f(q,u) \in \mathbb{R}$.

$$\left(d_i^{\hat{\psi}}((p,v))\right)(f) = \left[\partial_i \left(f \circ \hat{\psi}^{-1}\right)\right]((p,v))$$

For $i \in \{1,\dots,n\}$ we get derivatives w.r.t. "$\mathcal{M}$", $i \in \{n+1,\dots,2n\}$ w.r.t. "$T_p\mathcal{M}$".