Quantum Field Theory

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Mesoscopic Transport

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Course ???-????-??? of Prof. Dr. Oded Zilberberg

typesetting, compilation & editing by
T. Wolf

Abstract

Quantum mechanics has been extremely useful for the description of material properties. The rapid development in control and manipulation of coherent quantum systems in recent years allows for the study and utilization of coherent quantum phenomena as well as the exploration of quantum mechanical concepts in realistic many-body setups. In such devices there exists a subtle interplay between a plethora of effects such as, disorder, interactions, and out-of-equilibrium noise that compete and limit the regimes in which coherence survives. To describe the transport properties of such devices one requires more suitable tools that can take into account the interplay between these effects. In the proseminar, we shall review theoretical tools (both analytic and numerical) that have been developed to address such transport phenomena. These tools are actively used in contemporary research , are deeply linked to questions in quantum information theory, and are also an integral tool in the study of topological phenomena. We shall apply these methods to study realistic models.

In this proseminar, field theoretical methods for the description of many-body transport phenomena will be discussed.

Supervisors
Prof. Dr. O. Zilberberg
Dr. F. Herman
Dr. J. Tolsma
Dr. J. L. L. Villanueva
Dr. M. Biondi
Dr. W. Chen
Dr. C. Müller
Dr. E. Greplova
J. Shapiro
Dr. M. H. Fischer

Students
Lars Dehlwes
Daniel Puth
Zhongyu Zhang
Ludwig Hruza
Xiangzhou Zhu
Damian Moosbrugger
Mohsen Talebi
Guangze Chen
Jeremy Mann
Tim Fleischmann
Patrik Weber
Jürg Haag
Michael Denner
Siman Mathis
Renato Durrer
Guliuxin Jin
Lee Mang Hei Gordon
Anton Eder

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Chapter 1

Gaussian integrals & supermathematics

Anton Eder [1]

The diagrammatic perturbation approach from previous chapters is well suited for describing conductivity or other response functions, but becomes increasingly tedious for calculating higher order moments. Other phenomena like energy level statistics are completely inaccessible to this formalism. We thus need a new approach: Supersymmetric non-linear sigma models allow for efficient and mathematically rigorous calculation of higher order moments and level statistics. To derive such a model in a consistent way, we have to make use of two new concepts: Gaussian integrals and superlinear algebra. In this chapter, we will provide a short introduction to those mathematical tools, following mainly the book of Efetov [1] and notes by J. Shapiro [2].

1.1 Conductivity on a lattice

The arguably most important transport quantity of a solid is its conductivity σ. Since this chapter focuses on introducing the basic concepts necessary for deriving a non-linear sigma model for the conductivity, it is sufficient to look at σ on an infinite lattice. Although we are ultimately interested in the continuum version of σ, the lattice approach offers the advantage of simplifying most calculations while still capturing all relevant physical phenomena.

1.1.1 Discrete Kubo formula

Suppose that we have a system in equilibrium state ρ_0 and apply a weak perturbation

$$V(t) = -V_0 f(t) A$$

to it, where A is a Hermitian operator and the real function f accounts for the time dependence. To ensure that

$$\lim_{t \to -\infty} \rho(t) = \rho_0$$

[1] supervised by Jacob Shapiro

for the perturbed state $\rho(t)$, we require the perturbation to be turned on *adiabatically*, i.e.

$$\lim_{t\to-\infty} f(t) = 0 \,,$$
$$f(0) = 1 \,.$$

Linear response theory tells us that if we then probe this system with an observable B, its response to $V(t)$ can be described by

$$\begin{aligned} \langle B\rangle_{\rho(t=0)} &= \langle B\rangle_{\rho_0} + \chi_{BA} V_0 + \mathcal{O}(V_0^2) \,, \\ \chi_{BA} &= \mathrm{i}\int_{-\infty}^{0} \mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}Ht} B \mathrm{e}^{\mathrm{i}Ht}[A,\rho_0]\right) f(t)\,\mathrm{d}t \;. \end{aligned} \tag{1.1}$$

Since we are interested in the DC conductivity, we take our perturbation to be a weak electric field in x_j-direction

$$V(t) = \boldsymbol{E}\cdot\boldsymbol{x}\,\mathrm{e}^{\varepsilon t} = E_j x_j\,\mathrm{e}^{\varepsilon t}$$

in the limit $\varepsilon \to 0^+$ and probe the system by measuring the current density in x_i-direction

$$j_i = -\operatorname{Tr}\left(v_i\left(\rho(0)-\rho_0\right)\right) = \sigma_{ij}E_j + \mathcal{O}(E_j^2) \,,$$

with $\boldsymbol{v} = \mathrm{i}[H,\boldsymbol{x}]$ referring to the electron velocity. It should be noted that starting from this equation, we have to replace the regular trace $\mathrm{tr}(\,\cdot\,)$ by the *trace per unit volume*

$$\mathrm{Tr}(\,\cdot\,) := \lim_{\Lambda\to\mathbb{Z}^d} \frac{1}{|\Lambda|}\sum_{x\in\Lambda} \langle\delta_x|\;\cdot\;|\delta_x\rangle \;,$$

since x_k is not a trace class operator. However, all trace properties relevant for this derivation, i.e. linearity and cyclicity, still hold. Assuming that the equilibrium state corresponds to a Fermi-Dirac distribution at zero temperature and Fermi energy E, we can describe ρ_0 in the single-particle picture by the Fermi projection:

$$\rho_0 = \chi_{\leq E}(H) := \lim_{\beta\to\infty}\left(1+\mathrm{e}^{\beta(H-E)}\right)^{-1}$$

After inserting these expressions, equation (1.1) becomes

$$\sigma_{ij}(E) = -\lim_{\varepsilon\to0^+}\int_{-\infty}^{0} \mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}Ht}\,[H,x_i]\,\mathrm{e}^{\mathrm{i}Ht}\,[x_j,\chi_{\leq E}(H)]\right)\mathrm{e}^{\varepsilon t}\,\mathrm{d}t \;.$$

We can then rewrite

$$\mathrm{e}^{\varepsilon t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\varepsilon t}-1}{\varepsilon}\right) \,,$$

which allows us to perform a partial integration leading to

$$\sigma_{ij}(E) = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{0} \left(\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr}\left(\mathrm{e}^{-\mathrm{i}Ht} \left[H, x_i \right] \mathrm{e}^{\mathrm{i}Ht} \left[x_j, \chi_{\leq E}(H) \right] \right) \right) \frac{\mathrm{e}^{\varepsilon t} - 1}{\varepsilon} \, \mathrm{d}t \ . \tag{1.2}$$

For a derivation avoiding this point please see [3]. Making use of the cyclicity of the trace and the fact that $[H, \rho_0] = 0$, we can express the time derivative by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr}\left(\mathrm{e}^{-\mathrm{i}Ht} \left[H, x_i \right] \mathrm{e}^{\mathrm{i}Ht} \left[x_j, \chi_{\leq E}(H) \right] \right) &= \operatorname{Tr}\left(\left[H, x_i \right] \frac{\mathrm{d}}{\mathrm{d}t} \left[\mathrm{e}^{\mathrm{i}Ht} \, x_j \, \mathrm{e}^{-\mathrm{i}Ht}, \chi_{\leq E}(H) \right] \right) \\ &= \operatorname{Tr}\left(\left[H, x_i \right] \mathrm{e}^{\mathrm{i}Ht} \left[\mathrm{i}[H, x_j], \chi_{\leq E}(H) \right] \mathrm{e}^{-\mathrm{i}Ht} \right) \\ &= -\mathrm{i} \operatorname{Tr}\left(v_i \, \mathrm{e}^{\mathrm{i}Ht} \left[v_j, \chi_{\leq E}(H) \right] \mathrm{e}^{-\mathrm{i}Ht} \right) \ . \end{aligned}$$

We can use the spectral decomposition of H to express functions $f(H)$ as

$$f(H) = \int_{\lambda \in \mathbb{R}} f(\lambda) \, \mathrm{d}P(\lambda) \ ,$$

where P is the projection-valued spectral measure of H. This allows us to move the exponentials - and therefore the time dependence - out of the trace:

$$\begin{aligned} \operatorname{Tr}\left(v_i \, \mathrm{e}^{\mathrm{i}Ht} \left[v_j, \chi_{\leq E}(H) \right] \mathrm{e}^{-\mathrm{i}Ht} \right) &= \operatorname{Tr}\left(v_i \left(\mathrm{e}^{\mathrm{i}Ht} \, v_j \chi_{\leq E}(H) \, \mathrm{e}^{-\mathrm{i}Ht} - \mathrm{e}^{\mathrm{i}Ht} \, \chi_{\leq E}(H) v_j \, \mathrm{e}^{-\mathrm{i}Ht} \right) \right) \\ &= \int_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} \mathrm{e}^{\mathrm{i}(\lambda_1 - \lambda_2)t} \left(\chi_{\leq E}(\lambda_2) - \chi_{\leq E}(\lambda_1) \right) \operatorname{Tr}\left(v_i \, \mathrm{d}P(\lambda_1) \, v_j \, \mathrm{d}P(\lambda_2) \right) \end{aligned}$$

If we then rewrite

$$\mathrm{e}^{\varepsilon t} - 1 = t \int_0^{\varepsilon} \mathrm{e}^{\eta t} \, \mathrm{d}\eta \ ,$$

we can perform the time integration in equation (1.2):

$$\begin{aligned} \sigma_{ij}(E) &= \mathrm{i} \lim_{\varepsilon \to 0^+} \int_{\lambda_1, \lambda_2} \int_0^{\varepsilon} \frac{\mathrm{d}\eta}{\varepsilon} \left(\int_{-\infty}^{0} t \, \mathrm{e}^{\mathrm{i}(\lambda_1 - \lambda_2 - \mathrm{i}\eta)t} \, \mathrm{d}t \right) \left(\chi_{\leq E}(\lambda_1) - \chi_{\leq E}(\lambda_2) \right) \mathrm{d}m_{ij}(\lambda_1, \lambda_2) \\ &= \mathrm{i} \lim_{\varepsilon \to 0^+} \int_{\lambda_1, \lambda_2} \int_0^{\varepsilon} \frac{\mathrm{d}\eta}{\varepsilon} \frac{1}{(\lambda_1 - \lambda_2 - \mathrm{i}\eta)^2} \left(\chi_{\leq E}(\lambda_1) - \chi_{\leq E}(\lambda_2) \right) \mathrm{d}m_{ij}(\lambda_1, \lambda_2) \ , \end{aligned}$$

where we defined

$$\mathrm{d}m_{ij}(\lambda_1, \lambda_2) := \operatorname{Tr}\left(v_i \, \mathrm{d}P(\lambda_1) \, v_j \, \mathrm{d}P(\lambda_2) \right) \ .$$

For well-behaved functions f, we can replace

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^{\varepsilon} f(\eta) \, \mathrm{d}\eta = \lim_{\eta \to 0^+} f(\eta)$$

and with the help of the Sokhotski-Plemelj formula, we find that

$$\begin{aligned}\lim_{\eta\to 0^+}\frac{1}{(t-\mathrm{i}\eta)^2} &= -\frac{\mathrm{d}}{\mathrm{d}t}\lim_{\eta\to 0^+}\frac{1}{t-\mathrm{i}\eta}\\ &= -\left(\mathcal{P}\,\frac{1}{t}\right)' - i\pi\delta'(t)\,.\end{aligned}$$

In this equation, $\mathcal{P}$ refers to the Cauchy principal value. If we assume our system to be time-reversal invariant, one can easily show that

$$\mathrm{d}m_{ij}(\lambda_1,\lambda_2) = \mathrm{d}m_{ij}(\lambda_2,\lambda_1)\,.$$

The product $(\chi_{\leq E}(\lambda_1)-\chi_{\leq E}(\lambda_2))\,\mathrm{d}m_{ij}(\lambda_1,\lambda_2)$ is therefore an *odd* function of (λ_1,λ_2); it follows from symmetry considerations that the integration over the *even* $\mathcal{P}'$-term vanishes and we are left with integrating over the *odd* δ'-function:

$$\begin{aligned}\sigma_{ij}(E) &= \pi\int_{\lambda_1,\lambda_2}\delta'(\lambda_1-\lambda_2)\,(\chi_{\leq E}(\lambda_1)-\chi_{\leq E}(\lambda_2))\,\mathrm{d}m_{ij}(\lambda_1,\lambda_2)\\ &= \pi\int_{\lambda}(\partial_\lambda\,\chi_{\leq E}(\lambda))\,\mathrm{d}m_{ij}(\lambda,\lambda)\end{aligned}$$

At $T=0$, the Fermi distribution is simply a step function around E and we have

$$\partial_\lambda\,\chi_{\leq E}(\lambda) = \partial_\lambda\Theta(E-\lambda) = \delta(E-\lambda)$$

and thus

$$\begin{aligned}\sigma_{ij}(E) &= \pi\int_{\lambda}\delta(E-\lambda)\,\mathrm{Tr}\,(v_i\,\mathrm{d}P(\lambda)\,v_j\,\mathrm{d}P(\lambda))\\ &= \pi\,\partial_{\lambda_1}\partial_{\lambda_2}\,\mathrm{Tr}\,(v_i\,\chi_{\leq\lambda_1}(H)v_j\,\chi_{\leq\lambda_1}(H))\,\Big|_{\lambda_1=\lambda_2=E}\,.\end{aligned}\tag{1.3}$$

We can then make a connection to the Green's function by using Stone's formula (cf. page 237 of [4]) and express the density matrix as

$$\chi_{\leq E}(H) = \lim_{\eta\to 0^+}\frac{1}{\pi}\int_{-\infty}^{E}\mathrm{Im}[R(\epsilon+\mathrm{i}\eta)]\,\mathrm{d}\epsilon = \lim_{\eta\to 0^+}\frac{\eta}{\pi}\int_{-\infty}^{E}|R(\epsilon+\mathrm{i}\eta)|^2\,\mathrm{d}\epsilon\;,\tag{1.4}$$

where $\mathrm{Im}[O] := (O-\overline{O})/2\mathrm{i}$ and R refers to the resolvent of the Hamiltonian:

$$R(z) = (H-z\mathbb{1})^{-1} = \overline{R(\overline{z})}$$

Inserting equation (1.4) into the Kubo formula (1.3) and using the cyclic property of the trace, we obtain

$$\begin{aligned}\sigma_{ij}(E) &= \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \operatorname{Tr}\left(v_i\, \overline{R(E+\mathrm{i}\eta)} R(E+\mathrm{i}\eta)\, v_j\, R(E+\mathrm{i}\eta)\overline{R(E+\mathrm{i}\eta)}\right) \\ &= -\lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \operatorname{Tr}\left(R(E-\mathrm{i}\eta)[H,x_i]R(E-\mathrm{i}\eta)R(E+\mathrm{i}\eta)[H,x_j]R(E+\mathrm{i}\eta)\right) .\end{aligned}$$

With the useful identity

$$R[H,O]R = [O,R]$$

for any operator O, the above equation simplifies to

$$\sigma_{ij}(E) = -\lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \operatorname{Tr}\left(\left[x_i, R(E-\mathrm{i}\eta)\right]\left[x_j, R(E+\mathrm{i}\eta)\right]\right) .$$

At this point, we can apply Birkhoff's theorem and relate the trace per unit volume to the expectation value $\mathbb{E}[\,\cdot\,]$ of disorder averaging:

$$\begin{aligned}\sigma_{ij}(E) &= -\lim_{\eta\to 0^+} \lim_{\Lambda\to\mathbb{Z}^d} \frac{\eta^2}{\pi|\Lambda|} \sum_{x\in\Lambda} \langle\delta_x| \left[x_i, R(E-\mathrm{i}\eta)\right]\left[x_j, R(E+\mathrm{i}\eta)\right] |\delta_x\rangle \\ &= -\lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \mathbb{E}\left[\langle\delta_0| \left[x_i, R(E-\mathrm{i}\eta)\right]\left[x_j, R(E+\mathrm{i}\eta)\right] |\delta_0\rangle\right]\end{aligned}$$

Since $x_k |\delta_0\rangle = 0$, only one term in the expectation value remains:

$$\sigma_{ij}(E) = \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \mathbb{E}\left[\langle\delta_0| R(E-\mathrm{i}\eta) x_i x_j R(E+\mathrm{i}\eta) |\delta_0\rangle\right]$$

and by inserting an identity $\mathbb{1} = \sum_{x\in\mathbb{Z}^d} |\delta_x\rangle\langle\delta_x|$ between the two position operators, we arrive at

$$\begin{aligned}\sigma_{ij}(E) &= \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \mathbb{E}\left[\sum_{x\in\mathbb{Z}^d} x_i x_j \langle\delta_0| R(E-\mathrm{i}\eta) |\delta_x\rangle \langle\delta_x| R(E+\mathrm{i}\eta) |\delta_0\rangle\right] \\ &= \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \sum_{x\in\mathbb{Z}^d} x_i x_j\, \mathbb{E}\left[\overline{\langle\delta_x| R(E+\mathrm{i}\eta) |\delta_0\rangle} \langle\delta_x| R(E+\mathrm{i}\eta) |\delta_0\rangle\right] \qquad (1.5) \\ &= \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \sum_{x\in\mathbb{Z}^d} x_i x_j\, \mathbb{E}\left[|G(x,0;E+\mathrm{i}\eta)|^2\right]\end{aligned}$$

with the Green's function in position basis

$$G(x,y;z) := \langle\delta_x| (H - z\mathbb{1})^{-1} |\delta_y\rangle .$$

This is then our single particle, zero temperature Kubo formula for the DC conductivity on an infinite lattice. This formula and its derivation can also be found in Appendix A of [5].

1.1.2 Green's functions on a finite lattice

To simplify further calculations and get a better understanding of G, we now go to a *finite* lattice Λ, where operators become matrices and we can express the Green's function $G(x, y; E \pm \mathrm{i}\eta)$ as components of an inverse matrix

$$G_\Lambda(x, y; E \pm \mathrm{i}\eta) = \left(\frac{1}{H_\Lambda - (E \pm \mathrm{i}\eta)\mathbb{1}} \right)_{xy} .$$

We can connect our results for the finite lattice back to the Kubo formula (1.5) by using the fact that

$$\lim_{\Lambda \to \mathbb{Z}^d} G_\Lambda(x, y; z) = G(x, y; z) .$$

This is shown in [6]. Depending on the sign of the regulator, G_Λ is either a retarded ("$+\mathrm{i}\eta$") or advanced ("$-\mathrm{i}\eta$") Green's function. This can easily be seen by expressing the Green's function as a Laplace transform:

$$(H - (E \pm \mathrm{i}\eta))^{-1} = \mathrm{i} \int_0^{\pm\infty} \mathrm{e}^{-\mathrm{i}Ht} \mathrm{e}^{\mathrm{i}(E \pm \mathrm{i}\eta)t} \, \mathrm{d}t$$

We choose to model the electron dynamics by a simple Anderson Hamiltonian with a kinetic hopping term and a random potential diagonal in position space:

$$(H_\Lambda)_{xy} = T_{xy} + V_{xy} = \sum_{e \in \Lambda:\, |e|=1} \delta_{x-y,e} + v(x)\, \delta_{xy} , \tag{1.6}$$

where the potential values $v(x)$ are elements of an independent and identically distributed sequence $\{v(x)\}_{x \in \Lambda}$, each drawn with a probability distribution μ on $\mathbb{R}$ (cf. [6]). The main difficulty in evaluating the right-hand side of equation (1.5) now arises from having to average over the inverse of a random potential. However, after introducing both Gaussian integrals and superlinear algebra, we will be able to express G_Λ in a way that allows us to easily take the expectation value.

1.2 Path integrals

In the derivation of our non-linear sigma model, we will be making use of *Gaussian integrals*, which are closely related to *path integrals*. Although these two are not exactly the same, we wish to illustrate some physical principles using the latter before proceeding with the former. To do so, we will follow Chapter 9 of [7]. So far, all quantum calculations were done in the

Hamiltonian formalism, where the time evolution of a state is described by the operator

$$\hat{U} = \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right) .$$

The path integral formalism is an equivalent description that makes use of the system's classical Lagrangian instead of its quantum Hamiltonian. In this formalism, one can also easily see how classical mechanics arises from quantum physics.

1.2.1 The action principle in classical mechanics

In classical mechanics, the trajectory $\boldsymbol{q}$ of a particle going from point $\boldsymbol{x}_1 = \boldsymbol{q}(t_1)$ to $\boldsymbol{x}_2 = \boldsymbol{q}(t_2)$ is determined by the *action principle*: Along $\boldsymbol{q}(t)$, the action

$$S\left[\boldsymbol{q}\right] = \int_{t_1}^{t_2} L\left[\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t\right] \,\mathrm{d}t$$

becomes stationary, i.e. its variation vanishes:

$$\delta S\left[\boldsymbol{q}\right] = 0 \tag{1.7}$$

From this condition, we can derive the Euler-Lagrange equations and thus the equations of motion for any classical system, given that we know its Lagrangian L.

1.2.2 The action principle in quantum physics

As it turns out, one can generalize this to a quantum mechanical action principle. Since quantum theory is probabilistic, we now have to talk about transition amplitudes $K(\boldsymbol{x}_1, \boldsymbol{x}_2; t_2 - t_1)$ instead of deterministic trajectories $\boldsymbol{q}$. The probability of the particle going from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ in the time interval $t_2 - t_1$ is then given by $|K|^2$. In the path integral formalism, this amplitude is obtained by summing over all possible paths $\boldsymbol{q}(t)$ the particle could take going from $(t_1, \boldsymbol{x}_1)$ to $(t_2, \boldsymbol{x}_2)$, each weighted by a phase factor

$$\exp(\mathrm{i}\varphi[\boldsymbol{q}]) = \exp\left(\mathrm{i}\,\frac{S[\boldsymbol{q}]}{\hbar}\right) .$$

We thus obtain

$$K(\boldsymbol{x}_1, \boldsymbol{x}_2; t_2 - t_1) = \int \mathcal{D}\boldsymbol{q}\; \mathrm{e}^{\mathrm{i}S[\boldsymbol{q}]/\hbar} , \tag{1.8}$$

where the the right-hand side is a functional integration over all allowed paths $\boldsymbol{q}$ obeying the boundary conditions. In non-relativistic quantum mechanics, S is just the classical action

$$S[\boldsymbol{q}] = \int \left(\frac{m}{2}\dot{\boldsymbol{q}}^2 - V(\boldsymbol{q})\right) \mathrm{d}t \ .$$

In quantum field theory, on the other hand, we talk about fields instead of particles, so we have to replace trajectories $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^3$ by spacetime-dependent field configurations $\phi : \mathbb{R}^4 \to \mathbb{C}^n$; the action along a certain field configuration is then its spacetime-integrated Lagrangian density $\mathcal{L}$:

$$S[\phi] = \int_{\mathbb{R}^4} \mathcal{L}\left[\phi, \partial_\mu \phi, x_\mu\right] \mathrm{d}^4 x$$

The analog of equation (1.8) is therefore

$$K(\phi_1, \phi_2; T) = \langle \phi_2(\boldsymbol{x}, T) | \, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{H}T} \, |\phi_1(\boldsymbol{x}, 0)\rangle = \int \mathcal{D}\phi \, \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}^4 x \, \mathcal{L}[\phi]\right) \ .$$

1.2.3 Connection to classical mechanics

Taking the limit $\hbar \to 0$, the quantum action principle (1.8) becomes equivalent to the classical action principle (1.7). This can be understood intuitively: Paths close to the classical trajectory $\boldsymbol{q}_{\mathrm{cl}}$ all contribute with roughly the same phase φ_{cl}, since $\delta\varphi_{\mathrm{cl}} = \delta S[\boldsymbol{q}_{\mathrm{cl}}]/\hbar = 0$ by definition. The transition probability is thus enhanced by the constructive interference of those paths. Paths further away from $\boldsymbol{q}_{\mathrm{cl}}$, however, vary strongly in S and their phase contributions rotate with period $2\pi\hbar$. Therefore, when summing over those paths, their phases "average out" and one can neglect their contribution. For typical quantum systems, S has order of $\hbar$ and we have to consider many different paths. For classical systems, $S \gg \hbar$ and even slight deviations from $\boldsymbol{q}_{\mathrm{cl}}$ shift the action by $\delta S \gg \hbar$. Consequently, all paths not infinitesimally close to the classical one interfere destructively and the transition probability is determined entirely by the trajectory satisfying the action principle (1.7). More formally, one could also expand in powers of $\hbar$ around $\boldsymbol{q}_{\mathrm{cl}}$ and get quantum corrections corresponding to a WKB approximation.

1.3 Gaussian integrals

While the path integral formalism allows us to express matrix elements $\left(\mathrm{e}^{-\mathrm{i}tH}\right)_{xy}$ of the time evolution operator, the Kubo formula (1.5) requires the Green's function $\left((H - z\mathbb{1})^{-1}\right)_{xy}$. In this section, we will derive a way to express the matrix elements of such inverted operators

via Gaussian integrals. It can be easily shown that for $\varphi \in \mathbb{C}$ and any $a \in \mathbb{C}$ with $\mathrm{Re}[a] > 0$,

$$\frac{1}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-a|\varphi|^2} \, \mathrm{d}\varphi = \frac{1}{a} \, .$$

We are now interested in generalizing this formula for complex vectors $\varphi \in \mathbb{C}^n$ and matrices $A \in \mathbb{C}^{n \times n}$:

$$I_{\mathrm{B}}(A) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle} \, \mathrm{d}\varphi$$

To ensure convergence, A has to have a positive real part $\mathrm{Re}[A] > 0$. Let us also assume that A is unitarily diagonalizable:

$$A = U^\dagger \, \mathrm{diag}(a_1, \ldots, a_n) \, U$$

By substituting $\varphi \to U\tilde{\varphi}$ and using the fact that $|\det(U)| = 1$, we find that

$$\begin{aligned} I_{\mathrm{B}}(A) &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \tilde{\varphi}, \mathrm{diag}(a_1, \ldots, a_n)\tilde{\varphi} \rangle} \, \mathrm{d}\tilde{\varphi} \\ &= \prod_{i=1}^{n} \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{e}^{-a_i |\tilde{\varphi}_i|^2} \, \mathrm{d}\tilde{\varphi}_i \\ &= \prod_{i=1}^{n} \frac{1}{a_i} = \frac{1}{\prod_{i=1}^{n} a_i} \, . \end{aligned}$$

By definition of the determinant, we then have

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle} \, \mathrm{d}\varphi = \frac{1}{\det(A)} \, . \tag{1.9}$$

One can also show that this is invariant under shifts:

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi + \alpha, A(\varphi + \beta) \rangle} \, \mathrm{d}\varphi = \frac{1}{\det(A)} \quad \forall \alpha, \beta \in \mathbb{C}^n \tag{1.10}$$

By completing the square and using equation (1.10), we get

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle + \langle \alpha, \varphi \rangle + \langle \varphi, \beta \rangle} \, \mathrm{d}\varphi = \frac{\mathrm{e}^{\langle \alpha, A^{-1}\beta \rangle}}{\det(A)} \tag{1.11}$$

and have thus found the *partition function* of A with sources α and β, i.e. the *generating functional*

$$Z_{\mathrm{B}}(A; \alpha, \beta) := \frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle + \langle \alpha, \varphi \rangle + \langle \varphi, \beta \rangle} \, \mathrm{d}\varphi \, .$$

Differentiating both sides of equation (1.11) by components $\overline{\alpha_i}$ and β_j and subsequently evaluating them at $\alpha = \beta = 0$, we finally arrive at

$$\frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi_i \overline{\varphi_j}\, \mathrm{e}^{-\langle \varphi, A\varphi \rangle}\, \mathrm{d}\varphi = \frac{(A^{-1})_{ij}}{\det(A)} \,. \tag{1.12}$$

With this formula, we are now able to express the inverse of a matrix as a Gaussian integral, which could potentially simplify the disorder averaging in equation (1.5). However, we still have an additional factor of $\det(A)$, which can be highly non-trivial. By combining equations (1.9) and (1.12), we can eliminate this factor:

$$\left(A^{-1}\right)_{ij} = \frac{\int_{\mathbb{C}^n} \varphi_i \overline{\varphi_j}\, \mathrm{e}^{-\langle \varphi, A\varphi \rangle}\, \mathrm{d}\varphi}{\int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle}\, \mathrm{d}\varphi} = \frac{\partial_{\overline{\alpha_i}} \partial_{\beta_j} Z_{\mathrm{B}}(A; \alpha, \beta)}{Z_{\mathrm{B}}(A; 0, 0)} \bigg|_{\alpha=\beta=0}$$

Unfortunately, this normalization brings us back to the initial problem of having to average over the inverse of a random potential; we therefore have to find another way.

1.4 The replica trick

One possible way to do this would be the *replica trick*: Using equation (1.9), we can write

$$1 = \det(A) \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle}\, \frac{\mathrm{d}\varphi}{\pi^n}$$

and insert this into equation (1.12) $(N-1)$-times:

$$(A^{-1})_{ij} = \left(\det(A) \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle}\, \frac{\mathrm{d}\varphi}{\pi^n} \right)^{N-1} \det(A) \int_{\mathbb{C}^n} \varphi_i \overline{\varphi_j}\, \mathrm{e}^{-\langle \varphi, A\varphi \rangle}\, \frac{\mathrm{d}\varphi}{\pi^n}$$

With Fubini's theorem, we can write this as an integration over N identical replicas $\varphi^{(k)}$ of our system:

$$(A^{-1})_{ij} = \det(A)^N \int_{\mathbb{C}^{nN}} \varphi_i^{(1)} \overline{\varphi_j^{(1)}}\, \mathrm{e}^{-\sum_{k=1}^N \langle \varphi^{(k)}, A\varphi^{(k)} \rangle} \prod_{k=1}^N \frac{\mathrm{d}\varphi^{(k)}}{\pi^n} \quad \forall N \in \mathbb{N} \tag{1.13}$$

The *trick* is to now take the formal limit $N \to 0$:

$$(A^{-1})_{ij} = \lim_{N \to 0} \det(A)^N \int_{\mathbb{C}^{nN}} \varphi_i^{(1)} \overline{\varphi_j^{(1)}}\, \mathrm{e}^{-\sum_{k=1}^N \langle \varphi^{(k)}, A\varphi^{(k)} \rangle} \prod_{k=1}^N \frac{\mathrm{d}\varphi^{(k)}}{\pi^n}$$

If this limit were well-defined, we could take the limit of the first factor and thus eliminate the determinant:

$$(A^{-1})_{ij} = \lim_{N\to 0} \int_{\mathbb{C}^{nN}} \varphi_i^{(1)} \overline{\varphi_j^{(1)}}\, \mathrm{e}^{-\sum_{k=1}^N \langle \varphi^{(k)}, A\varphi^{(k)}\rangle} \prod_{k=1}^N \frac{\mathrm{d}\varphi^{(k)}}{\pi^n}$$

If we were also allowed to interchange the order of disorder averaging and taking the limit, we would have found a way to easily calculate the expectation value:

$$\mathbb{E}\left[(A^{-1})_{ij}\right] = \lim_{N\to 0} \mathbb{E}\left[\int_{\mathbb{C}^{nN}} \varphi_i^{(1)} \overline{\varphi_j^{(1)}}\, \mathrm{e}^{-\sum_{k=1}^N \langle \varphi^{(k)}, A\varphi^{(k)}\rangle} \prod_{k=1}^N \frac{\mathrm{d}\varphi^{(k)}}{\pi^n}\right]$$

However, since the right-hand side of equation (1.13) is only defined for $N \in \mathbb{N}$, it is not clear how the continuous limit $N \to 0$ can be understood. It is also not obvious that $\mathbb{E}[\,\cdot\,]$ and $\lim_{N\to 0}$ can be interchanged. The replica trick is therefore not mathematically rigorous and can in fact lead to wrong results; for more detailed criticism, see [8]. A rigorous alternative to this trick is the *supersymmetry method*, which we will now derive.

1.5 Fermionic integrals

In the previous section, we integrated over complex vectors $\varphi \in \mathbb{C}^n$, whose components φ_x could be interpreted as the values of some (scalar) field at each lattice point $x \in \Lambda$. Since those components are regular *commuting* complex numbers, i.e.

$$\varphi_i \varphi_j = \varphi_j \varphi_i\,,$$

such a field would be of *bosonic* nature. How could we then describe *fermionic* fields? To see this, we have to introduce the concept of anti-commuting variables.

1.5.1 Grassmann numbers

An n-dimensional complex vector space V with an orthonormal basis $\{e_i\}_{i=1}^n$ can be regarded as a $2n$-dimensional real vector space $\tilde{V}$ with a complex structure

$$J : \tilde{V} \to \tilde{V}, \quad J^2 = -\mathbb{1}$$

and an orthonormal basis $\{e_i, Je_i\}_{i=1}^n$. We can use this to define the Grassmannian algebra

$$\mathcal{G}^n := \mathbb{C} \otimes_{\mathbb{R}} \bigwedge \tilde{V}$$

as the complexification of $\bigwedge \tilde{V}$, the 2^{2n}-dimensional exterior algebra over $\tilde{V}$. As such, it has complex dimension 2^{2n} and $2n$ complex generators $\{e_i, Je_i\}_{i=1}^n$. Since we are ultimately interested in describing fermions, it is useful to make a change of basis to

$$\psi_i := \frac{1}{\sqrt{2}} \left(1 \otimes e_i - \mathrm{i} \otimes Je_i\right) ,$$
$$\tilde{\psi}_i := \frac{1}{\sqrt{2}} \left(1 \otimes e_i + \mathrm{i} \otimes Je_i\right) .$$

This way, we get a natural particle anti-particle structure on $\mathcal{G}^n$; both ψ_i and $\tilde{\psi}_i$ are eigenvectors of J and related to each other via complex conjugation:

$$\overline{\psi_i} = \tilde{\psi}_i \ , \quad \overline{\tilde{\psi}_i} = \psi_i$$

To simplify further definitions, let us label the generators of $\mathcal{G}^n$ more compactly by

$$\{\gamma_i\}_{i=1}^{2n} := \left\{\psi_i, \tilde{\psi}_i\right\}_{i=1}^{n} .$$

A general element of $\mathcal{G}^n$ can then be written as

$$\eta = \sum_{j_1,\ldots,j_{2n}\in\{0,1\}} \alpha_{j_1\ldots j_{2n}} \gamma_1^{j_1} \wedge \ldots \wedge \gamma_{2n}^{j_{2n}} \quad \text{with} \quad \alpha_{j_1\ldots j_{2n}} \in \mathbb{C} \, , \tag{1.14}$$

where $\wedge$ denotes the exterior product, i.e. the multiplication operation on $\mathcal{G}^n$. A more detailed characterization of $\mathcal{G}^n$ and its structure can be found in Folland's book [9]; for our basic introduction, however, it is sufficient to know that the generators of $\mathcal{G}^n$ *anti-commute*,

$$\gamma_i \wedge \gamma_j = -\gamma_j \wedge \gamma_i \, , \tag{1.15}$$

making them an obvious choice for representing *fermionic* fields. In particular, this means that their square always vanishes:

$$\gamma_i^2 := \gamma_i \wedge \gamma_i = 0$$

As a consequence, any analytic function of these so-called *Grassmann variables* is a *finite* linear combination of all 2^{2n} basis elements of $\mathcal{G}^n$, given that we interpret the powers in the series representation of such a function as exterior products:

$$\eta^k := \eta^{\wedge k} = \bigwedge_{i=1}^{k} \eta \ , \quad \eta \in \mathcal{G}^n$$

For our purposes, it is also convenient to define both differentiation and integration on this algebra as linear functionals mapping $\mathcal{G}^n$ to $\mathbb{C}$. Without having to deal with any kind of

limit, we can define an algebraic derivative ∂_{γ_i} by

$$\partial_{\gamma_i} 1 := 0\,,$$
$$\partial_{\gamma_i}\gamma_j := \delta_{ij}$$

and extend this linearly via the generalized Leibniz rule

$$\partial_{\gamma_i}(\eta_1 \wedge \eta_2) := (\partial_{\gamma_i}\eta_1) \wedge \eta_2 + (-1)^{\deg(\eta_1)}\eta_1 \wedge (\partial_{\gamma_i}\eta_2) \tag{1.16}$$

to higher-degree elements as defined in equation (1.14). We also define integration to be the same as differentiation:

$$\int \cdot\ \mathrm{d}\gamma_i := \partial_{\gamma_i} \tag{1.17}$$

Changing the notation back from (γ_i, γ_{i+1}) to $(\psi_i, \tilde{\psi}_i)$, we define:

$$\begin{aligned}\int_{\mathcal{G}^n} \cdot\ \mathrm{d}\psi\,\mathrm{d}\tilde{\psi} &:= \int_{\mathcal{G}^n} \cdot\ \mathrm{d}\psi_1 \wedge \mathrm{d}\tilde{\psi}_1 \wedge \ldots \wedge \mathrm{d}\psi_n \wedge \mathrm{d}\tilde{\psi}_n \\ &= \partial_{\tilde{\psi}_n}\partial_{\psi_n}\ldots\partial_{\tilde{\psi}_1}\partial_{\psi_1}\end{aligned}$$

This can be interpreted as the integration over a fermionic field $\psi = (\psi_1, \ldots, \psi_n)$ and its corresponding antiparticle field $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_n)$ and we find, for example, that

$$\int_{\mathcal{G}^n} 1\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi} = 0 \quad \text{and} \tag{1.18}$$

$$\int_{\mathcal{G}^n} \psi_1 \wedge \tilde{\psi}_1 \wedge \ldots \wedge \psi_n \wedge \tilde{\psi}_n\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi} = 1\,. \tag{1.19}$$

There also exists a fermionic analog for Fubini's theorem, which can be easily shown with definitions (1.17) and (1.16):

$$\begin{aligned}&\int_{\mathcal{G}^n} F(\gamma_1, \ldots, \gamma_l) \wedge G(\gamma_{l+1}, \ldots, \gamma_{2n})\,\mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_{2n} \\ &= (-1)^{(2n-l-1)\deg(\partial_{\gamma_l}\ldots\partial_{\gamma_1}F(\gamma_1,\ldots,\gamma_l))}\left(\int_{\mathcal{G}^{l/2}} F(\gamma_1, \ldots, \gamma_l)\,\mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_l\right) \\ &\quad \cdot \left(\int_{\mathcal{G}^{n-l/2}} G(\gamma_{l+1}, \ldots, \gamma_{2n})\,\mathrm{d}\gamma_{l+1} \ldots \mathrm{d}\gamma_{2n}\right)\end{aligned}$$

for functions $F : \mathcal{G}^{l/2} \to \mathcal{G}^n$ and $G : \mathcal{G}^{n-l/2} \to \mathcal{G}^n$. We can also generalize Fubini's theorem to integrals over both bosonic and fermionic variables: For a function $f : X \to \mathcal{G}^n$ mapping some measure space X with measure μ to the Grassmannian algebra $\mathcal{G}^n$, the integral

$$\int_{x\in X}\left(\int_{\mathcal{G}^n} f(x)\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi}\right)\mathrm{d}\mu(x)$$

has a clear interpretation, since

$$\int_{\mathcal{G}^n} f(x)\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi} \in \mathbb{C} \quad \forall x \in X\ .$$

On the other hand, we may expand any such f in a basis $\{e_i\}_{i=1}^{2^{2n}}$ of $\mathcal{G}^n \cong \mathbb{C}^{2^{2n}}$:

$$f(x) = \sum_{i=1}^{2^{2n}} f_i(x) e_i$$

with coordinate maps $f_i : X \to \mathbb{C}$. Viewing $\mathcal{G}^n$ as a Banach space, we can interpret

$$\int_{x\in X} f(x)\,\mathrm{d}\mu(x) = \sum_{i=1}^{2^{2n}} \left(\int_{x\in X} f_i(x)\,\mathrm{d}\mu(x)\right) e_i \in \mathcal{G}^n$$

as a Bochner integral. The integrals on the right-hand side of this equation are then obviously well-defined. We can thus conclude that

$$\int_{\mathcal{G}^n} \left(\int_{x\in X} f(x)\,\mathrm{d}\mu(x)\right) \mathrm{d}\psi\,\mathrm{d}\tilde{\psi} = \int_{x\in X} \left(\int_{\mathcal{G}^n} f(x)\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi}\right) \mathrm{d}\mu(x)$$

as long as f is Bochner-integrable.

1.5.2 Fermionic Gaussians

We are now interested in the fermionic equivalent of equation (1.9):

$$I_{\mathrm{F}}(A) := \int_{\mathcal{G}^n} \mathrm{e}^{-\langle\psi, A\psi\rangle}\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi}$$

After defining the bilinear form

$$\langle\psi, A\psi\rangle := \sum_{i,j=1}^{n} A_{ij}\,\tilde{\psi}_i \wedge \psi_j\ ,$$

we can expand the Gaussian up to n-th order (all higher orders vanish because of the anti-commutativity (1.15)):

$$\mathrm{e}^{-\langle\psi, A\psi\rangle} = \sum_{k=0}^{n} \frac{1}{k!} \left(\sum_{i,j=1}^{n} A_{ij}\,\tilde{\psi}_i \wedge \psi_j\right)^{\wedge k}$$

Integrating over ψ and $\tilde{\psi}$, all terms of order smaller than n in the Gaussian also vanish (cf. equation (1.18)) and the integral simplifies to

$$\begin{aligned} I_{\mathrm{F}}(A) &= \int_{\mathcal{G}^n} \frac{1}{n!} \left(\sum_{i,j=1}^{n} A_{ij}\, \tilde{\psi}_i \wedge \psi_j \right)^{\wedge n} \mathrm{d}\psi\, \mathrm{d}\tilde{\psi} \\ &= \int_{\mathcal{G}^n} \frac{(-1)^n}{n!} \sum_{i_1, j_1, \ldots, i_n, j_n = 1}^{n} A_{i_1 j_1} \ldots A_{i_n j_n}\, \tilde{\psi}_{i_1} \wedge \psi_{j_1} \ldots \tilde{\psi}_{i_n} \wedge \psi_{j_n}\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi} \end{aligned}$$

Commuting all $\tilde{\psi}_{i_k} \wedge \psi_{j_k}$ to $-\psi_{j_k} \wedge \tilde{\psi}_{i_k}$ and summing over all permutations of i_k, we get an additional factor of $((-1)^n\, n!)$:

$$I_{\mathrm{F}}(A) = \int_{\mathcal{G}^n} \sum_{j_1, \ldots, j_n = 1}^{n} A_{1 j_1} \ldots A_{n j_n}\, \psi_1 \wedge \tilde{\psi}_{j_1} \wedge \ldots \wedge \psi_n \wedge \tilde{\psi}_{j_n}\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi}$$

We then permute the Grassmann variables into the "right" order, giving each term in the sum an additional sign:

$$\begin{aligned} I_{\mathrm{F}}(A) &= \int_{\mathcal{G}^n} \left(\sum_{j_1, \ldots, j_n = 1}^{n} \operatorname{sgn}(\sigma_j) A_{1 j_1} \ldots A_{n j_n} \right) \psi_1 \wedge \tilde{\psi}_1 \ldots \psi_n \wedge \tilde{\psi}_n\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi} \\ &= \det(A) \int_{\mathcal{G}^n} \psi_1 \wedge \tilde{\psi}_1 \wedge \ldots \wedge \psi_n \wedge \tilde{\psi}_n\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi} \end{aligned}$$

Using equation (1.19), we then finally arrive at our fermionic Gaussian integral:

$$\int_{\mathcal{G}^n} \mathrm{e}^{-\langle \psi, A\psi \rangle}\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi} = \det(A) \tag{1.20}$$

Comparing this result to equation (1.9), we can see that we again obtain the determinant of A, but this time in the numerator! Analogous to the bosonic case (1.10), one can then show that for any $\eta \in \mathcal{G}^n$,

$$\int_{\mathcal{G}^n} \mathrm{e}^{-\langle \psi, A\psi \rangle + \langle \eta, \psi \rangle + \langle \psi, \eta \rangle}\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi} = \det(A)\, \mathrm{e}^{\langle \eta, A^{-1}\eta \rangle}\ . \tag{1.21}$$

We have thus found the *fermionic partition function* of A with source η:

$$Z_{\mathrm{F}}(A; \eta) := \int_{\mathcal{G}^n} \mathrm{e}^{-\langle \psi, A\psi \rangle + 2\,\mathrm{Re}[\langle \eta, \psi \rangle]}\, \mathrm{d}\psi\, \mathrm{d}\tilde{\psi}$$

After differentiating both sides of equation (1.21) by $\tilde{\eta}_i$ and η_j and evaluating the result at $\eta = 0$, we find the fermionic equivalent to equation (1.12):

$$\int_{\mathcal{G}^n} \psi_i \wedge \tilde{\psi}_j \, \mathrm{e}^{-\langle \psi, A\psi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi} = \det(A) \, (A^{-1})_{ij} \tag{1.22}$$

To get rid of the determinant factor, we could normalize this expression by inserting equation (1.20):

$$\left(A^{-1}\right)_{ij} = \frac{\int_{\mathcal{G}^n} \psi_i \wedge \tilde{\psi}_j \, \mathrm{e}^{-\langle \psi, A\psi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi}}{\int_{\mathcal{G}^n} \mathrm{e}^{-\langle \psi, A\psi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi}} = \left. \frac{\partial_{\tilde{\eta}_i} \partial_{\eta_j} Z_{\mathrm{F}}(A; \eta)}{Z_{\mathrm{F}}(A; 0)} \right|_{\eta = 0}$$

Since this once again brings us back to our inital problem of having to average over the denominator, we have to find a smarter way to eliminate the determinant, for example by integrating over both bosonic *and* fermionic Gaussians. Combining either equations (1.12) and (1.20) or (1.9) and (1.22), we get

$$\begin{aligned} \left(A^{-1}\right)_{ij} &= \left(\int_{\mathcal{G}^n} \mathrm{e}^{-\langle \psi, A\psi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi} \right) \left(\frac{1}{\pi^n} \int_{\mathbb{C}^n} \varphi_i \overline{\varphi_j} \, \mathrm{e}^{-\langle \varphi, A\varphi \rangle} \, \mathrm{d}\varphi \right) \\ &= \left(\frac{1}{\pi^n} \int_{\mathbb{C}^n} \mathrm{e}^{-\langle \varphi, A\varphi \rangle} \, \mathrm{d}\varphi \right) \left(\int_{\mathcal{G}^n} \psi_i \wedge \tilde{\psi}_j \, \mathrm{e}^{-\langle \psi, A\psi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi} \right) . \end{aligned}$$

With Fubini's theorem for mixed integrals, this can be written as

$$\begin{aligned} \left(A^{-1}\right)_{ij} &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathcal{G}^n} \varphi_i \overline{\varphi_j} \, \mathrm{e}^{-\langle \psi, A\psi \rangle - \langle \varphi, A\varphi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi} \, \mathrm{d}\varphi \\ &= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \int_{\mathcal{G}^n} \psi_i \wedge \tilde{\psi}_j \, \mathrm{e}^{-\langle \psi, A\psi \rangle - \langle \varphi, A\varphi \rangle} \, \mathrm{d}\psi \, \mathrm{d}\tilde{\psi} \, \mathrm{d}\varphi \ . \end{aligned} \tag{1.23}$$

We have thus found a way to overcome our "denominator problem"; to simplify further calculations, however, we have to introduce another new concept.

1.6 Superlinear algebra

Looking at equation (1.23), one can immediately see that it would be very convenient to somehow combine the two fields φ and ψ into a single new field. Indeed, one can define a so-called *supervector*

$$\Phi = \varphi \oplus \psi \in \mathbb{C}^n \oplus \mathcal{G}^n$$

with both bosonic and fermionic components. This is a formal object. Since such supervectors are elements of a highly non-trivial space, we should first examine how concepts from regular linear algebra can be translated to these superspaces.

1.6.1 Linear algebra with anti-commuting scalars

This section follows Chapter 2 of Efetov's book [1]; we will therefore momentarily switch to his notation, which differs slightly from our previous conventions. Most importantly, Efetov defines complex conjugation in such a way that

$$\overline{(\overline{\psi})} = -\psi \ ,$$

while in our convention,

$$\overline{(\overline{\psi})} = \overline{\tilde{\psi}} = \psi \ .$$

Given supervectors of the form

$$\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \ ,$$

a general *supermatrix* F acting on such vectors can be written in the following block form:

$$F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$$

Here, a and b are "regular" matrices mapping bosons to bosons and fermions to fermions, while σ and ρ map fermions to bosons and vice versa. It follows, then, that a and b must have commuting components, whereas σ and ρ must have anti-commuting components. We define the *supertranspose* of such a matrix by

$$F^{\mathrm{T}} := \begin{pmatrix} a^{\mathrm{t}} & -\rho^{\mathrm{t}} \\ \sigma^{\mathrm{t}} & b^{\mathrm{t}} \end{pmatrix}$$

with $\cdot^{\mathrm{t}}$ referring to regular transposition $(A^{\mathrm{t}})_{ij} = A_{ji}$. This way, it is ensured that

$$(F_1 F_2)^{\mathrm{T}} = F_2^{\mathrm{T}} F_1^{\mathrm{T}} \ .$$

We should note that in general, $(F^{\mathrm{T}})^{\mathrm{T}} \neq F$; however, defining Hermitian conjugation of supermatrices by

$$F^{\dagger} := \overline{F^{\mathrm{T}}} \ ,$$

one can easily see that

$$(F_1 F_2)^{\dagger} = F_2^{\dagger} F_1^{\dagger} \ ,$$
$$(F^{\dagger})^{\dagger} = F \ .$$

Taking into account the anticommutativity of the fermionic components of Φ, we define a generalized *supertrace*

$$\mathrm{str}(F) := \mathrm{tr}(a) - \mathrm{tr}(b)$$

that inherits all important properties of the regular trace, e.g. cyclicity:

$$\mathrm{str}(F_1 \dots F_{n-1} F_n) = \mathrm{str}(F_n F_1 \dots F_{n-1})$$

Similarly, we define a *superdeterminant*

$$\mathrm{sdet}(F) := \det\big(a - \sigma b^{-1} \rho\big) \det\big(b^{-1}\big) \ ,$$

which again inherits all properties from its regular variant, e.g. multiplicativity:

$$\mathrm{sdet}(F_1 F_2) = \mathrm{sdet}(F_1)\ \mathrm{sdet}(F_2)$$

We then also maintain the following useful identity:

$$\log(\mathrm{sdet}(F)) = \mathrm{str}(\log(F))$$

Finally, defining the Hermitian conjugate of supervectors by

$$\Phi^\dagger = \overline{\Phi^{\mathrm{t}}} := \big(\overline{\varphi}, \overline{\psi}\big) \ ,$$

one can also define their scalar product

$$\big\langle \Phi^1, \Phi^2 \big\rangle := \Phi^{1\dagger} \Phi^2 = \sum_{i=1}^{2n} \overline{\Phi_i^1} \Phi_i^2 = \sum_{i=1}^{n} \Big(\overline{\varphi_i^1} \varphi_i^2 + \tilde{\psi}_i^1 \wedge \psi_i^2 \Big)$$

and bilinear forms

$$\begin{aligned} \big\langle \Phi^1, F\Phi^2 \big\rangle := \Phi^{1\dagger} F \Phi^2 &= \sum_{i,j=1}^{2n} \overline{\Phi_i^1} F_{ij} \Phi_j^2 \\ &= \sum_{i,j=1}^{n} \Big(\overline{\varphi_i^1} a_{ij} \varphi_j^2 + \overline{\varphi_i^1} \sigma_{ij} \wedge \psi_j^2 + \tilde{\psi}_i^1 \wedge \rho_{ij} \varphi_j^2 + \tilde{\psi}_i^1 \wedge b_{ij} \psi_j^2 \Big) \end{aligned}$$

in the usual manner. These few definitions suffice for our calculations; a more detailed introduction to superlinear algebra can be found in [1].

1.6.2 Gaussian superintegrals

Let us now return to equation (1.23) and our previous notation. Interpreting A as a supermatrix $A \oplus A$, we can condense the two bilinear forms into one:

$$\langle \Phi, A\Phi \rangle = \langle \varphi, A\varphi \rangle + \langle \psi, A\psi \rangle$$

We are also free to define the integration measure on our superspace as

$$\mathrm{d}\Phi := \frac{\mathrm{d}\varphi\,\mathrm{d}\psi\,\mathrm{d}\tilde{\psi}}{\pi^n}$$

and can thus write our inversion formula in the compact form

$$\left(A^{-1}\right)_{ij} = \int \Phi_i^\alpha \overline{\Phi_j^\alpha}\, \mathrm{e}^{-\langle \Phi, A\Phi\rangle}\, \mathrm{d}\Phi \ ,$$

where the index $\alpha \in \{1,2\}$ refers to either the bosonic or fermionic part of Φ and is completely arbitrary (cf. equation (1.23)). Note that there is no implied summation over α. One could then interpret this integral as a field average over a Gaussian distribution determined by the supermatrix A, suggesting the shorthand notation

$$\langle\, \cdot\, \rangle_A := \int \cdot\ \mathrm{e}^{-\langle \Phi, A\Phi\rangle}\, \mathrm{d}\Phi$$

such that we can compactly write

$$\begin{aligned} \langle 1 \rangle_A &= 1 \ , \\ \left\langle \Phi_i^\alpha \overline{\Phi_j^\alpha} \right\rangle_A &= (A^{-1})_{ij} \ . \end{aligned} \tag{1.24}$$

1.7 Taking the disorder average

After introducing these concepts, we can now return to our initial problem of calculating the conductivity. Equipped with the supersymmetric inversion formula (1.24), we can express the electronic Green's functions as

$$\begin{aligned} G_\Lambda(x,y;E \pm \mathrm{i}\eta) &= \left([H_\Lambda - (E \pm \mathrm{i}\eta)\mathbb{1}]^{-1}\right)_{xy} \\ &= \pm\mathrm{i}\left([\eta\mathbb{1} \pm \mathrm{i}(H_\Lambda - E\mathbb{1})]^{-1}\right)_{xy} \\ &= \pm\mathrm{i}\left\langle \Phi_x^\alpha \overline{\Phi_y^\alpha} \right\rangle_{\eta\mathbb{1} \pm \mathrm{i}(H_\Lambda - E\mathbb{1})} \ . \end{aligned}$$

Note that it's necessary to factor out $\pm\mathrm{i}$ to ensure convergence of the integral; since the regulator η is by definition a positive number, this always works. Inserting this back into the expectation value in the Kubo formula (1.5), we find:

$$\begin{aligned} \mathbb{E}\left[|G_\Lambda(x,0;E+\mathrm{i}\eta)|^2\right] &= \mathbb{E}\left[\overline{G_\Lambda(x,0;E+\mathrm{i}\eta)}\ G_\Lambda(x,0;E+\mathrm{i}\eta)\right] \\ &= \mathbb{E}\left[G_\Lambda(0,x;E-\mathrm{i}\eta)\ G_\Lambda(x,0;E+\mathrm{i}\eta)\right] \\ &= \mathbb{E}\left[\left\langle \Phi_0^\alpha \overline{\Phi_x^\alpha} \right\rangle_{\eta\mathbb{1} - \mathrm{i}(H_\Lambda - E\mathbb{1})} \left\langle \Phi_x^\alpha \overline{\Phi_0^\alpha} \right\rangle_{\eta\mathbb{1} + \mathrm{i}(H_\Lambda - E\mathbb{1})}\right] \end{aligned} \tag{1.25}$$

With the superintegral version of Fubini's theorem, we can then combine these two integrals into one:

$$\begin{aligned}
&\left\langle \Phi_0^\alpha \overline{\Phi_x^\alpha} \right\rangle_{\eta\mathbb{1}-\mathrm{i}(H_\Lambda - E\mathbb{1})} \left\langle \Phi_x^\alpha \overline{\Phi_0^\alpha} \right\rangle_{\eta\mathbb{1}+\mathrm{i}(H_\Lambda - E\mathbb{1})} =: \left\langle \Phi_0^\alpha \overline{\Phi_x^\alpha} \right\rangle_{F^-} \left\langle \Phi_x^\alpha \overline{\Phi_0^\alpha} \right\rangle_{F^+} \\
&= \left(\int \Phi_0^{\alpha-} \overline{\Phi_x^{\alpha-}} \mathrm{e}^{-\left\langle \Phi^-, F^- \Phi^- \right\rangle} \, \mathrm{d}\Phi^- \right) \left(\int \Phi_x^{\alpha+} \overline{\Phi_0^{\alpha+}} \mathrm{e}^{-\left\langle \Phi^+, F^+ \Phi^+ \right\rangle} \, \mathrm{d}\Phi^+ \right) \\
&= \int \Phi_0^{\alpha-} \overline{\Phi_x^{\alpha-}} \Phi_x^{\alpha+} \overline{\Phi_0^{\alpha+}} \mathrm{e}^{-\left\langle \Phi^-, F^- \Phi^- \right\rangle - \left\langle \Phi^+, F^+ \Phi^+ \right\rangle} \, \mathrm{d}\Phi^- \, \mathrm{d}\Phi^+ \\
&= \int \Psi_0^{\alpha-} \overline{\Psi_x^{\alpha-}} \Psi_x^{\alpha+} \overline{\Psi_0^{\alpha+}} \mathrm{e}^{-\langle \Psi, F\Psi \rangle} \, \mathrm{d}\Psi = \left\langle \Psi_0^{\alpha-} \overline{\Psi_x^{\alpha-}} \Psi_x^{\alpha+} \overline{\Psi_0^{\alpha+}} \right\rangle_F \, ,
\end{aligned}$$

where we defined a new supervector

$$\Psi := \Phi^- \oplus \Phi^+$$

and a new supermatrix

$$F := F^- \oplus F^+ := [\eta\mathbb{1} - \mathrm{i}(H_\Lambda - E\mathbb{1})] \oplus [\eta\mathbb{1} + \mathrm{i}(H_\Lambda - E\mathbb{1})]$$

such that we can express the modulus square of the Green's function as a single superfield average. Introducing the supermatrix

$$\Lambda := \sigma_3 \otimes \mathbb{1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

(which should not to be confused with the lattice Λ) we can compactly write

$$F = \eta\mathbb{1} - \mathrm{i}\Lambda \otimes (H_\Lambda - E\mathbb{1}) \, .$$

Going back to the impurity average (1.25), one can show that the expectation value and the superintegration $\int \mathrm{d}\Psi$ can be swapped:

$$\begin{aligned}
\mathbb{E}\left[|G_\Lambda(x,0;E+\mathrm{i}\eta)|^2 \right] &= \mathbb{E}\left[\left\langle \Psi_0^{\alpha-} \overline{\Psi_x^{\alpha-}} \Psi_x^{\alpha+} \overline{\Psi_0^{\alpha+}} \right\rangle_F \right] \\
&= \mathbb{E}\left[\int \Psi_0^{\alpha-} \overline{\Psi_x^{\alpha-}} \Psi_x^{\alpha+} \overline{\Psi_0^{\alpha+}} \mathrm{e}^{-\langle \Psi, F\Psi \rangle} \, \mathrm{d}\Psi \right] \\
&= \int \mathbb{E}\left[\Psi_0^{\alpha-} \overline{\Psi_x^{\alpha-}} \Psi_x^{\alpha+} \overline{\Psi_0^{\alpha+}} \mathrm{e}^{-\langle \Psi, F\Psi \rangle} \right] \mathrm{d}\Psi
\end{aligned}$$

and since the superfield components are non-random, we can move them out of the expectation value:

$$\mathbb{E}\left[|G_\Lambda(x,0;E+\mathrm{i}\eta)|^2 \right] = \int \Psi_0^{\alpha-} \overline{\Psi_x^{\alpha-}} \Psi_x^{\alpha+} \overline{\Psi_0^{\alpha+}} \, \mathbb{E}\left[\mathrm{e}^{-\langle \Psi, F\Psi \rangle} \right] \mathrm{d}\Psi \, .$$

We are thus left with the impurity average over the Gaussian; writing out the Hamiltonian (1.6), we find:

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\langle\Psi,F\Psi\rangle}\right] &= \mathbb{E}\left[\mathrm{e}^{-\langle\Psi,[\eta\mathbb{1}-\mathrm{i}\Lambda\otimes(H_\Lambda-E\mathbb{1})]\Psi\rangle}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-\langle\Psi,[\eta\mathbb{1}-\mathrm{i}\Lambda\otimes(T_\Lambda-E\mathbb{1})]\Psi\rangle}\ \mathrm{e}^{\mathrm{i}\langle\Psi,(\Lambda\otimes V_\Lambda)\Psi\rangle}\right]\end{aligned}$$

The randomness of the Hamiltonian is contained entirely within the potential term V_Λ; it can therefore be shown that we can also move the non-random Gaussian out of the expectation value:

$$\mathbb{E}\left[\mathrm{e}^{-\langle\Psi,F\Psi\rangle}\right] = \mathrm{e}^{-\langle\Psi,[\eta\mathbb{1}-\mathrm{i}\Lambda\otimes(T_\Lambda-E\mathbb{1})]\Psi\rangle}\ \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\Psi,(\Lambda\otimes V_\Lambda)\Psi\rangle}\right]$$

Using the fact that the potential is diagonal in position basis (cf. equation (1.6)) and the potential values $\{v(x)\}_{x\in\Lambda}$ are *independently* distributed across the lattice sites, we can write

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\Psi,(\Lambda\otimes V_\Lambda)\Psi\rangle}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{x\in\Lambda}v(x)\langle\Psi_x,\Lambda\Psi_x\rangle}\right] \\ &= \prod_{x\in\Lambda}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}v(x)\langle\Psi_x,\Lambda\Psi_x\rangle}\right] =: \prod_{x\in\Lambda}h_x\left(\langle\Psi_x,\Lambda\Psi_x\rangle\right)\ ,\end{aligned} \tag{1.26}$$

where h_x denotes the characteristic function

$$h_x(t) := \int_{v\in\mathbb{R}}\mathrm{e}^{\mathrm{i}vt}\,\mathrm{d}\mu(v) \equiv \mathbb{E}\left[\mathrm{e}^{\mathrm{i}v(x)t}\right]\ \forall x$$

of the distribution μ according to which the potential values $\{v(x)\}_{x\in\Lambda}$ are distributed. Assuming for simplicity that μ is a Gaussian distribution, we find

$$\begin{aligned}h_x(t) &= \int_{v\in\mathbb{R}}\mathrm{e}^{\mathrm{i}vt}\left(\frac{\mathrm{e}^{-\frac{1}{2\sigma_x^2}v^2}}{\sqrt{2\pi}\sigma_x}\,\mathrm{d}v\right) \\ &= \int_{v\in\mathbb{R}}\mathrm{e}^{-\frac{\sigma_x^2}{2}t^2}\left(\frac{\mathrm{e}^{-\frac{1}{2\sigma_x^2}(v-\mathrm{i}\sigma_x^2t)^2}}{\sqrt{2\pi}\sigma_x}\,\mathrm{d}v\right) \\ &= \mathrm{e}^{-\frac{\sigma_x^2}{2}t^2}\int_{v\in\mathbb{R}}\mathrm{d}\mu(v-\mathrm{i}\sigma_x^2t) = \mathrm{e}^{-\frac{\sigma_x^2}{2}t^2}\end{aligned}$$

and our expectation value (1.26) can be written as

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\Psi,(\Lambda\otimes V_\Lambda)\Psi\rangle}\right] &= \prod_{x\in\Lambda}\mathrm{e}^{-\frac{\sigma_x^2}{2}\langle\Psi_x,\Lambda\Psi_x\rangle^2} \\ &= \mathrm{e}^{-\sum_{x\in\Lambda}\frac{\sigma_x^2}{2}\langle\Psi_x,\Lambda\Psi_x\rangle^2}\ .\end{aligned}$$

Since the potential values are also *identically* distributed, $\sigma_x = \sigma\ \forall x$ and we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle\Psi,(\Lambda\otimes V_\Lambda)\Psi\rangle}\right] &= \mathrm{e}^{-\frac{\sigma^2}{2}\sum_{x\in\Lambda}\langle\Psi_x,\Lambda\Psi_x\rangle^2} \\ &= \mathrm{e}^{-\frac{\sigma^2}{2}\langle\Psi,\Lambda\Psi\rangle^2}\ .\end{aligned}$$

Finally, we can insert this back into the Kubo formula (1.5):

$$\begin{aligned}\sigma_{ij}(E) &= \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \sum_{\boldsymbol{x}\in\mathbb{Z}^d} x_i x_j \lim_{\Lambda\to\mathbb{Z}^d} \mathbb{E}\left[|G_\Lambda(x,0;E+\mathrm{i}\eta)|^2\right] \\ &= \lim_{\eta\to 0^+} \frac{\eta^2}{\pi} \sum_{\boldsymbol{x}\in\mathbb{Z}^d} x_i x_j \lim_{\Lambda\to\mathbb{Z}^d} \int \Psi_0^{\alpha-}\overline{\Psi_{\boldsymbol{x}}^{\alpha-}}\Psi_{\boldsymbol{x}}^{\alpha+}\overline{\Psi_0^{\alpha+}} \mathrm{e}^{-\mathcal{F}_\Lambda[\Psi]}\,\mathrm{d}\Psi \ ,\end{aligned}$$

where we defined an "effective action"

$$\begin{aligned}\mathcal{F}_\Lambda[\Psi] &:= -\log\left(\mathbb{E}\left[\mathrm{e}^{-\langle\Psi,F\Psi\rangle}\right]\right) \\ &= \langle\Psi,\left[\eta\mathbb{1}-\mathrm{i}\Lambda\otimes(T_\Lambda-E\mathbb{1})\right]\Psi\rangle + \frac{\sigma^2}{2}\langle\Psi,\Lambda\Psi\rangle^2 \ ,\end{aligned} \tag{1.27}$$

going back to the notion of path integrals. We were thus able to perform the disorder averaging at the cost of introducing a ϕ^4-type self-interaction of our superfield Ψ. To make efficient calculations in the presence of this term, we will have to introduce the concept of non-linear sigma models, which will be the focus of chapter **??**.

1.8 From the lattice to the continuum

This last section is devoted to making the transition from the lattice to the continuum, where our superfields become continuous functions of spacetime, matrices become linear operators and integrals over Ψ become functional integrals:

$$\begin{aligned}(\Psi_x)_{x\in\Lambda} &\to \Psi(\boldsymbol{r}) \\ (H_{xy})_{x,y\in\Lambda} &\to \hat{H} \\ \int \mathrm{d}\Psi &\to \int \mathcal{D}\Psi\end{aligned}$$

Our Gaussian integrals thus change into

$$\langle\,\cdot\,\rangle_F = \int \cdot\ \mathrm{e}^{-\langle\Psi,F\Psi\rangle}\,\mathrm{d}\Psi \to \int \cdot\ \mathrm{e}^{-\int \mathrm{d}\boldsymbol{x}\Psi^\dagger\hat{F}\Psi}\,\mathcal{D}\Psi \ . \tag{1.28}$$

We will also adopt the notation of Efetov and introduce a conjugate field

$$\overline{\Psi} := \Psi^\dagger\Lambda \ ,$$

which should not be confused with complex conjugate of Ψ. We note that the product

$$\overline{\Psi}\Psi = \Psi^\dagger\Lambda\Psi = \Psi^{-\dagger}\Psi^- - \Psi^{+\dagger}\Psi^+$$

is *not* positive definite. Modelling the impurities by a white noise potential, our effective action for the DC conductivity then has the following form:

$$\mathcal{F}_{\mathrm{DC}}[\Psi] = \int \mathrm{d}\boldsymbol{x} \left\{ \eta \Psi^\dagger \Psi - \mathrm{i}\overline{\Psi} \left(\frac{\hat{p}^2}{2m} - E \right) \Psi + \frac{\sigma^2}{2} \left(\overline{\Psi}\Psi \right)^2 \right\}$$

As we can see, this expression is a direct analog to the lattice action (1.27). To make the physics more interesting and establish a connection to Efetov, we want to generalize this action for the AC conductivity. As we have seen many times before, this means we have to introduce an additional frequency dependence to the advanced Green's function:

$$G(x, y; E - \mathrm{i}\eta)\, G(x, y; E + \mathrm{i}\eta) \to G(x, y; E + \omega - \mathrm{i}\eta)\, G(x, y; E + \mathrm{i}\eta)$$

We can account for this by adding a projector

$$P^- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \Lambda)$$

onto the space of the advanced superfield Ψ^- to our superoperator $\hat{F}$ in equation (1.28):

$$\hat{F}_{\mathrm{DC}} = \begin{pmatrix} \eta - \mathrm{i}(\hat{H} - E) & 0 \\ 0 & \eta + \mathrm{i}(\hat{H} - E) \end{pmatrix} = \eta - \mathrm{i}\Lambda \otimes (\hat{H} - E)$$

$$\to \hat{F}_{\mathrm{AC}} = \begin{pmatrix} \eta - \mathrm{i}(\hat{H} - E + \omega) & 0 \\ 0 & \eta + \mathrm{i}(\hat{H} - E \end{pmatrix} = \eta - \mathrm{i}\left[\Lambda \otimes (\hat{H} - E) + \frac{\omega}{2}(\mathbb{1} + \Lambda) \right]$$

The effective action for the AC conductivity thus becomes

$$\mathcal{F}_{\mathrm{AC}}[\Psi] = \int \mathrm{d}\boldsymbol{x} \left\{ \eta \Psi^\dagger \Psi - \mathrm{i}\overline{\Psi} \left(\frac{\hat{p}^2}{2m} - E + \frac{\omega}{2}(\mathbb{1} + \Lambda) \right) \Psi + \frac{\sigma^2}{2} \left(\overline{\Psi}\Psi \right)^2 \right\} .$$

1.9 Conclusion

Introducing various new concepts like path integrals, Grassmann numbers and supervectors, we were able to express the conductivity of a system as a functional integral over a weighted Gaussian. In contrast to the diagrammatic approach from previous chapters, this allowed us to non-perturbatively calculate the disorder average of σ, albeit at the cost of introducing a ϕ^4-type interaction to our theory, again forcing us to use approximations. To deal with this interaction efficiently, one can use *non-linear sigma models*, which will be introduced in the next chapter.

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