Gaussian Integrals

1 $A \in M_n(\mathbb{R})$; $A = A^T$, A is pos. def.

Claim: $\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)\, dx = \frac{\pi^{n/2}}{\sqrt{\det(A)}}$

Proof: Bcs. $A > 0$, $\exists\, A^{-1} \in M_n(\mathbb{R})$ s.t. $A^{-1} > 0$. $(A^{-1})^T = A^{-1}$ also.
also symmetric
A pos. def. matrix has a unique sq. root, call it $\sqrt{A^{-1}}$.
Make a change of var. $v := \sqrt{A^{-1}}^{-1} x \Rightarrow x = \sqrt{A^{-1}}\, v$

Then $\int_{\mathbb{R}^n} \exp(-\langle \sqrt{A^{-1}} v, A\sqrt{A^{-1}} v\rangle)\, |\det(\sqrt{A^{-1}})|\, dv =$

$$= |\det(\sqrt{A^{-1}})| \int_{\mathbb{R}^n} \exp(-\langle v, \underbrace{\sqrt{A^{-1}} A \sqrt{A^{-1}}}_{\mathbb{1}} v\rangle)\, dv$$

$$= |\det(\sqrt{A^{-1}})| \int_{\mathbb{R}^n} \exp(-\|v\|^2)\, dv$$

$$= |\det(\sqrt{A^{-1}})| \int_{\mathbb{R}^n} \underbrace{\exp\left(-\sum_{j=1}^n v_j^2\right)}_{\prod_{j=1}^n \exp(-v_j^2)} dv$$

$$\overset{\text{Fubini}}{=} |\det(\sqrt{A^{-1}})| \prod_{j=1}^n \underbrace{\int_{\mathbb{R}} \exp(-v_j^2)\, dv_j}_{\sqrt{\pi}}$$

(by computing square of quantity and using polar coordinates)

$$= \frac{\sqrt{\pi}^{\,n}}{\sqrt{\det(A)}}$$

Let $C \in M_n(\mathbb{C})$: $C^* = C$ $\wedge$ $0 \notin \sigma(C)$ $(\det(C) \neq 0)$

Claim: $\int_{\mathbb{C}^n} \exp(-\langle z, Cz\rangle)\, dz = \frac{\pi^n}{\det(C)}$

Proof: C is Hermitian $\Rightarrow$ It is diagonalizable
real & non-z
$\Leftrightarrow \exists\, U \in M_n(\mathbb{C})$ unitary w/ $UCU^{-1} = \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$
Make a change of var. $w := U^{-1} z$ to obtain

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$$\int_{\mathbb{C}^n} \exp(-\langle z, Cz\rangle)\,dz = \int_{\mathbb{C}^n} \exp(-\langle \mathcal{U}w, C\mathcal{U}w\rangle)\, \underbrace{|\det(\mathcal{U}^{-1})|}_{|e^{i\varphi}|=1,\ \text{unitarity}}\, dw =$$

$\mathcal{U}$ unitary

$$\overset{\mathcal{U}\text{ unitary}}{=} \int_{\mathbb{C}^n} \exp(-\langle w, \Lambda w\rangle)\,dw$$

$$= \int_{\mathbb{C}^n} \exp\Big(-\sum_{j=1}^n \bar{w}_j \lambda_j w_j\Big)\,dw$$

$$= \int_{\mathbb{C}^n} \prod_{j=1}^n \exp(-\bar{w}_j \lambda_j w_j)\,dw_j$$

Fubini

$$\overset{\text{Fubini}}{=} \prod_{j=1}^n \int_{\mathbb{C}} \exp(-\bar{w}_j \lambda_j w_j)\,dw_j$$

$w_j := x_j + iy_j$

And
$$\int_{\mathbb{C}} \exp(-\bar{w}_j \lambda_j w_j)\,dw_j = \int_{\mathbb{R}^2} \exp\big(-\underbrace{(x_j - iy_j)\lambda_j(x_j+iy_j)}_{-(\lambda_j x_j^2 + \lambda_j y_j^2 + \cancel{i\lambda_j x_j y_j} - \cancel{i\lambda_j y_j x_j})}\big)\,dx_j\,dy_j$$

$$= \int_{\mathbb{R}^2} \exp(-\lambda_j(x_j^2 + y_j^2))\,dx_j\,dy_j$$

Fubini

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \exp(-\lambda_j x_j^2)\,dx_j \int_{\mathbb{R}} \exp(-\lambda_j y_j^2)\,dy_j$$

$\tilde{x}_j := \sqrt{\lambda_j}\,x_j$

$$= \int_{\mathbb{R}} \exp(-\tilde{x}_j^2)\frac{1}{\sqrt{\lambda_j}}\,d\tilde{x}_j \;\ldots \text{same} \ldots$$

$$= \frac{\pi}{\lambda_j}$$

$\Rightarrow$ For the whole integral we thus get:

$$\prod_{j=1}^n \frac{\pi}{\lambda_j} = \frac{\pi^n}{\prod_{j=1}^n \lambda_j} \underset{\text{det invar. under unitaries.}}{=} \frac{\pi^n}{\det(C)}$$

Let $A \in M_n(\mathbb{R})$: $A = A^T$, A pos. def.

Claim: $\displaystyle\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle + \langle \omega, x\rangle)\, dx = \frac{\sqrt{\pi}^n}{\sqrt{\det(A)}} \exp(-\tfrac{1}{4}\langle \omega, A^{-1}\omega\rangle)$

Proof: A is symmetric and real

$\Rightarrow \exists R \in O(n)$: $RAR^{-1} = \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

Make a change of variables $y := R^{-1}x$ to get

$$\int_{\mathbb{R}^n} \exp(-\underbrace{\langle Ry, ARy\rangle}_{\langle y, \Lambda y\rangle} + \underbrace{\langle \omega, Ry\rangle}_{\langle R^{-1}\omega, y\rangle})\, \underbrace{|\det(R^{-1})|}_{|\pm 1| = 1 \text{ orthogonality}} dy =$$

$$-\langle y, \Lambda y\rangle + \langle R^{-1}\omega, y\rangle = \sum_{j=1}^{n} -y_j \lambda_j y_j + (R^{-1}\omega)_j y_j$$

$$= \sum_{j=1}^{n} -\lambda_j \underbrace{\left(y_j^2 - 2\frac{(R^{-1}\omega)_j}{2\lambda_j} y_j\right)}_{\left(y_j - \frac{(R^{-1}\omega)_j}{2\lambda_j}\right)^2 - \left(\frac{(R^{-1}\omega)_j}{2\lambda_j}\right)^2}$$

So that we get for the integral (w/ Fubini):

$$\exp\left(+\sum_{j=1}^{n} \frac{(R^{-1}\omega)_j^2}{4\lambda_j}\right) \prod_{j=1}^{n} \underbrace{\int_{\mathbb{R}} \exp\left(-\lambda_j\left(y_j - \frac{(R^{-1}\omega)_j}{2\lambda_j}\right)^2\right) dy_j}_{\text{transl. invariant}}$$

$$\Rightarrow \quad = \frac{\sqrt{\pi}}{\sqrt{\lambda_j}}$$

$$= \exp\left(\sum_{j=1}^{n} \frac{(R^{-1}\omega)_j^2}{4\lambda_j}\right) \frac{\sqrt{\pi}^n}{\sqrt{\det(A)}}$$

But $\displaystyle\sum_{j=1}^{n} \frac{(R^{-1}\omega)_j^2}{\lambda_j} = \langle R^{-1}\omega, \Lambda^{-1} R^{-1}\omega\rangle = \langle \omega, R\Lambda^{-1}R^{-1}\omega\rangle$

$$= \langle \omega, A^{-1}\omega\rangle$$

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Let $(x_f, x_i) \in \mathbb{R}^2$ be given. $(t_i, t_f) \in \mathbb{R}^2$ arbitrary, $\hbar \equiv 1$, $m \equiv 1$

By the path integral formulation of QM, we know:

$$\langle e^{+iHt_f}\delta_{x_f}, e^{-iHt_i}\delta_{x_i}\rangle = \int_{\left\{ \begin{smallmatrix} q:\mathbb{R}\to\mathbb{R} \\ t\mapsto x(t) \end{smallmatrix} \middle| \begin{smallmatrix} q(t_i)=x_i \\ q(t_f)=x_f \end{smallmatrix} \right\}} \mathcal{D}q \, \exp\left(i\int_{t_i}^{t_f}(L(q))(t)\,dt\right)$$

where H is the Ham. assoc. w/ the sys. & L is the corresponding Lagrangian: $H = T+V$

$L = T-V$

Formally H is the Legendre transf. of L:

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

We are given $V(q) := \frac{1}{2}w^2 q^2 - F\cdot q$

for some $w \in \mathbb{R}$, $F: \mathbb{R}\to\mathbb{R}$

Def.: $\tilde{F}(E) := \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-iEt} F(t)\,dt$ (Fourier transf. of F)

$$H = \frac{1}{2}\dot{q}^2 + \frac{1}{2}w^2 q^2 - F\cdot q$$

Define $H_0 := \frac{1}{2}\dot{q}^2 + \frac{1}{2}w^2 q^2$

I. Claim: $$\lim_{\substack{t_i\to-\infty \\ t_f\to+\infty}} \langle e^{+iHt_f}\delta_{x_f}, e^{-iHt_i}\delta_{x_i}\rangle = \left(\lim_{\substack{t_i\to-\infty \\ t_f\to+\infty}} \langle e^{+iH_0t_f}\delta_{x_f}, e^{-iH_0t_i}\delta_{x_i}\rangle\right) \times$$

$$\left(\lim_{\varepsilon\to0^+} \exp\left(-\frac{i}{2}\int \frac{\tilde{F}(E)\tilde{F}(-E)}{E^2-w^2+i\varepsilon}\,dE\right)\right)$$

Proof: Note that $\int_{-\infty}^{\infty}(L(q))(t)\,dt$ may not necessarily converge, so that $\exp\left(i\int_{-\infty}^{\infty}(L(q))(t)\,dt\right)$ may oscillate and we wouldn't get a convergent path integral.

The way to "cure" this is to write $w^2 = \lim_{\varepsilon\to0^+} w^2 - i\varepsilon$ and "pretend" that we may pull out $\lim_{\varepsilon\to0^+}$ out of both integrals (if they converge uniformly in ε, perhaps...) so that we obtain:

$$\int_{-\infty}^{\infty}\left[\frac{1}{2}\dot{q}^2-\frac{1}{2}(\omega^2-i\varepsilon)q^2+Fq\right]dt =$$

$$= S(q) + \underbrace{\int_{-\infty}^{\infty}\frac{1}{2}i\varepsilon q^2\,dt}_{\frac{1}{2}i\varepsilon\int_{-\infty}^{\infty}q^2\,dt}$$

positive & big

So that the path integrand is

$$e^{iS(q)} \rightsquigarrow e^{iS(q)-\varepsilon\infty}$$ which hopefully converges.

Thus we have:

$$\lim_{\substack{t_f\to\infty\\ t_i\to-\infty}}\langle e^{+iHt_f}\delta_{x_f}, e^{-iHt_i}\delta_{x_i}\rangle =$$

$$= \lim_{\substack{t_f\to\infty\\ t_i\to-\infty}}\lim_{\varepsilon\to0^+}\int \mathcal{D}q\; e^{i\int_{t_i}^{t_f}\left(\frac{1}{2}\dot{q}^2-\frac{1}{2}\omega^2q^2+Fq\right)dt-\varepsilon\int_{t_i}^{t_f}\frac{1}{2}q^2\,dt}$$

$\tilde{\omega}^2 := \omega^2 - i\varepsilon$

$$= \lim_{\varepsilon\to0^+}\int \mathcal{D}q\; e^{i\int_{-\infty}^{\infty}\left(\frac{1}{2}\dot{q}^2-\frac{1}{2}\tilde{\omega}^2q^2+Fq\right)dt}$$

Write $q(t) := \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{+iEt}\,\tilde{q}(E)\,dE$

$$\Rightarrow \dot{q}(t) = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{iEt}\,iE\,\tilde{q}(E)\,dE$$

$$\begin{aligned}\Rightarrow S(q) = \int_{-\infty}^{\infty}\Bigg[&\frac{1}{2}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{iEt}\,iE\,\tilde{q}(E)\,dE\right)\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\tilde{E}t}\,i\tilde{E}\,\tilde{q}(\tilde{E})\,d\tilde{E}\right) - \\ &-\frac{1}{2}\tilde{\omega}^2\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{iEt}\,\tilde{q}(E)\,dE\right)\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\tilde{E}t}\,\tilde{q}(\tilde{E})\,d\tilde{E}\right) + \\ &+\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{iEt}\,\tilde{F}(E)\,dE\right)\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{i\tilde{E}t}\,\tilde{q}(\tilde{E})\,d\tilde{E}\right)\Bigg]dt\end{aligned}$$

Fubini

$\int_{-\infty}^{\infty}e^{i(E+\tilde{E})t}\,dt = 2\pi\delta(E+\tilde{E})$

$\int\frac{1}{\sqrt{2\pi}}\cdots\underset{2\pi}{\delta(E+\tilde{E})}\,d\tilde{E} = \sqrt{2\pi}\cdots$

$$= \int_{-\infty}^{\infty}\frac{\cancel{\sqrt{2\pi}}}{\cancel{\sqrt{2\pi}}}\underbrace{\Big(\underbrace{\frac{1}{2}iE\,\tilde{q}(E)\,i(-E)\,\tilde{q}(-E)}_{E^2\tilde{q}(E)\tilde{q}(-E)}-\frac{1}{2}\tilde{\omega}^2\,\tilde{q}(E)\,\tilde{q}(-E)}_{\frac{1}{2}(E^2-\tilde{\omega}^2)\tilde{q}(E)\tilde{q}(-E)}+\tilde{F}(E)\,\tilde{q}(-E)\Big)\,dE$$

Next, note that $\displaystyle\int_{-\infty}^{\infty}\tilde F(E)\tilde q(-E)\,dE \underset{E\mapsto -E}{=} \int_{+\infty}^{-\infty}\tilde F(-E)\tilde q(E)\,(-dE) = \int_{-\infty}^{\infty}\tilde F(-E)\tilde q(+E)\,dE$

$$\Rightarrow \int_{-\infty}^{\infty}\tilde F(E)\tilde q(-E)\,dE = \int_{-\infty}^{\infty}\frac{1}{2}\left(\tilde F(E)\tilde q(-E)+\tilde F(-E)\tilde q(E)\right)dE$$

$$\Rightarrow S(q) = \frac{1}{2}\int_{-\infty}^{\infty}\left[\tilde q(E)(E^2-\tilde\omega^2)\tilde q(-E)+\tilde F(E)\tilde q(-E)+\tilde F(-E)\tilde q(E)\right]dE$$

Next note that the path integral is, by definition, translation invariant, where by translation we mean shift path by any other path (the shift-path does <u>not</u> vary during integration $\equiv$ const).

Also note that now we are integrating over paths $\tilde q$ and not q.

Make the shift

$$\left(E\longmapsto \hat q(E)\right)\longmapsto\left(E\longmapsto \tilde q(E)-\frac{\tilde F(E)}{E^2-\tilde\omega^2}\right)$$

path $E\mapsto \frac{\hat F(E)}{E^2-\tilde\omega^2}$ does not vary as we vary $\hat q$.

Then we obtain:

$$S\left(\tilde q-\frac{\tilde F(\cdot)}{\cdot^2-\tilde\omega^2}\right) = \frac{1}{2}\int_{-\infty}^{\infty}\Bigg[\left(\tilde q(E)-\frac{\tilde F(E)}{E^2-\tilde\omega^2}\right)(E^2-\tilde\omega^2)\left(\tilde q(-E)-\frac{\tilde F(-E)}{E^2-\tilde\omega^2}\right)+$$

$$+\tilde F(E)\left(\tilde q(-E)-\frac{\tilde F(-E)}{E^2-\tilde\omega^2}\right)+\tilde F(-E)\left(\hat q(E)-\frac{\tilde F(E)}{E^2-\tilde\omega^2}\right)\Bigg]dE =$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\Bigg[\tilde q(E)(E^2-\tilde\omega^2)\tilde q(-E)\cancel{-\tilde q(E)\tilde F(-E)}\cancel{-\tilde F(E)\tilde q(-E)}+\cancel{\frac{\tilde F(E)\tilde F(-E)}{E^2-\tilde\omega^2}}$$

$$\cancel{+\tilde F(E)\tilde q(-E)}\cancel{-\frac{\tilde F(E)\tilde F(-E)}{E^2-\tilde\omega^2}}\cancel{+\tilde F(-E)\tilde q(E)}-\frac{\tilde F(-E)\tilde F(E)}{E^2-\tilde\omega^2}\Bigg]dE =$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}\left[\tilde q(E)(E^2-\tilde\omega^2)\tilde q(-E)-\frac{\tilde F(-E)\tilde F(E)}{E^2-\tilde\omega^2}\right]dE$$

Plug this back into the path integral to obtain:

$$\lim_{\varepsilon\to 0^+}\int\mathcal{D}\tilde q\,\exp\left(i\frac{1}{2}\int_{-\infty}^{\infty}\left[\tilde q(E)(E^2-\tilde\omega^2)\tilde q(-E)-\frac{\tilde F(-E)\tilde F(E)}{E^2-\tilde\omega^2}\right]dE\right)$$

Now use the fact that $\tilde F$ does not depend on $\tilde q$ (it is a fixed path) so that it may be taken out of the path integral, to obtain:

replaced back $\tilde\omega \mapsto \omega$.

$$\left[\lim_{\varepsilon\to 0^+} \exp\left(-\frac{i}{2}\int_{\mathbb{R}} \frac{\tilde F(E)\tilde F(-E)}{E^2-\tilde\omega^2}\,dE\right)\right] \underbrace{\int \mathcal{D}\tilde q \, \exp\left(i\frac{1}{2}\int_{-\infty}^{\infty} \tilde q(E)(E^2-\omega^2)\tilde q(-E)\,dE\right)}_{=\lim_{\substack{t_i\to-\infty\\ t_f\to+\infty}} \langle e^{-iH_0 t_f}\delta_{x_f},\, e^{-iH_0 t_i}\delta_{x_i}\rangle}$$

II.

$$\lim_{\varepsilon\to 0^+} \exp\left(-\frac{i}{2}\int_{\mathbb{R}} \frac{\tilde F(E)\tilde F(-E)}{E^2-\tilde\omega^2}\,dE\right) = \lim_{\varepsilon\to 0^+} \exp\left(-\int_{\mathbb{R}} \frac{i}{2(E^2-\tilde\omega^2)} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-iEt} F(t)\,dt \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{+iEs} F(s)\,ds\, dE\right)$$

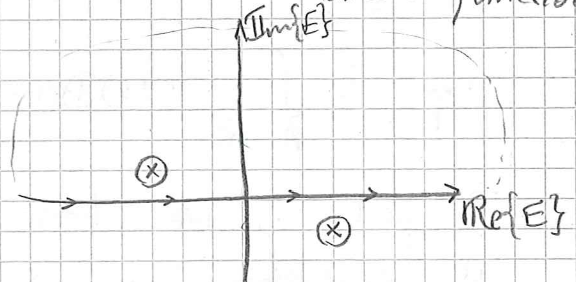
$$= \lim_{\varepsilon\to 0^+} \exp\left(-\frac{i}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} e^{-iE(t-s)} \frac{F(t)F(s)}{E^2-\omega^2+i\varepsilon}\, dE\,dt\,ds\right)$$

$$= \lim_{\varepsilon\to 0^+} \exp\left(-\frac{i}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(t)F(s) \underbrace{\left(\int_{\mathbb{R}} \frac{1}{2\pi} \frac{e^{-iE(t-s)}}{E^2-\omega^2+i\varepsilon}\,dE\right)}_{D(t-s)} dt\,ds\right)$$

Fourier transform of Green's function

Can calculate explicitly:

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{e^{-iEt}}{E^2-\omega^2+i\varepsilon}\,dE$$



$$E^2-\omega^2+i\varepsilon = 0$$

$$E = \pm\sqrt{\omega^2-i\varepsilon} = \pm\omega \mp i\varepsilon$$

Depending on the sign of t, we choose the upper or lower semi-circle to get that the integral on the semi-circle is zero.
If $t>0$, pick lower, so $-iEt$ has a negative real part: lower pole.
If $t<0$, pick upper, so $-iEt$ has a negative real part: upper pole.

Now compute the residues of the poles:

$$\frac{1}{E^2-\omega^2+i\varepsilon} = \frac{1}{(E+\omega-i\varepsilon)(E-\omega+i\varepsilon)} \Rightarrow$$ simple poles; residues are just the factors.

Thus at the upper pole we get

$$R(-\omega+i\tilde{\varepsilon}) = \frac{\exp(-i(-\omega+i\tilde{\varepsilon})t)}{((-\omega+i\tilde{\varepsilon})-\omega+i\tilde{\varepsilon})}$$

$$\overset{\tilde{\varepsilon}\to 0}{=} \frac{e^{i\omega t}}{-2\omega}$$

$$R(+\omega-i\tilde{\varepsilon}) = \frac{\exp(-i(+\omega-i\tilde{\varepsilon})t)}{((+\omega-i\tilde{\varepsilon})+\omega-i\tilde{\varepsilon})}$$

$$\overset{\tilde{\varepsilon}\to 0}{=} \frac{e^{-i\omega t}}{+2\omega}$$

$$\Rightarrow \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{1}{2\pi} \frac{e^{-iEt}}{E^2-\omega^2+i\varepsilon}\, dE = \frac{1}{2\omega} \begin{cases} -e^{i\omega t} & t<0 \\ -e^{-i\omega t} & t>0 \end{cases}$$

extra minus sign comes from fact contour was clockwise

$$= -\frac{1}{2\omega} e^{-i\omega|t|}$$

$$\Rightarrow \boxed{D(t-s) = -\frac{1}{2\omega} e^{-i\omega|t-s|}}$$

Finally we obtain:

$$\exp\left(-\frac{i}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(t)D(t-s)F(s)\, dt\, ds\right) =: Q$$

In part I. we have found Q is the ratio between the transition amp. $x_i \to x_f$ in time ∞, between the case w/ F and the case w/o F.

Note that $\mathbb{1} = \int_{\mathbb{R}} \delta_x \langle \delta_x, \cdot \rangle\, dx$, so that even though we were working w/ initial & final states given by δ_x, we could actually build any initial/final vector:

$$\frac{\lim\limits_{\substack{t_f\to\infty \\ t_i\to-\infty}} \langle e^{+iHt_f}\psi_f, e^{-iHt_i}\psi_i\rangle}{\lim\limits_{\substack{t_f\to\infty \\ t_i\to-\infty}} \langle e^{+iH_0t_f}\psi_f, e^{-iH_0t_i}\psi_i\rangle} = \frac{\lim\limits_{t} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\langle \delta_x, \psi_f\rangle}\langle \delta_y, \psi_i\rangle \langle e^{+iHt_f}\delta_x, e^{-iHt_i}\delta_y\rangle\, dx\,dy}{\lim\limits_{t} \quad \ldots}$$

$$= \frac{\lim\limits_{t} \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \overline{\langle \delta_x, v_p\rangle} \langle \delta_y, v_q\rangle \langle e^{-iH_0 t_f}\delta_x, e^{-iH_0 t_i}\delta_y\rangle e^{-\frac{i}{2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} F(t)D(t-s)F(s)dt\,ds}\, dx\,dy}{\lim\limits_{t} \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \overline{\langle \delta_x, v_p\rangle} \langle \delta_y, v_q\rangle \langle e^{-iH_0 t_f}\delta_x, e^{-iH_0 t_i}\delta_y\rangle\, dx\,dy}$$

$$= \exp\left(-\frac{i}{2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} F(t)D(t-s)F(s)\,dt\,ds\right)$$

III. Def.: $W_E[F] := \int \mathcal{D}q \, \exp\left(-\int \left[\frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2}\omega^2 q^2 + F(\tau)q(\tau)\right] d\tau\right)$

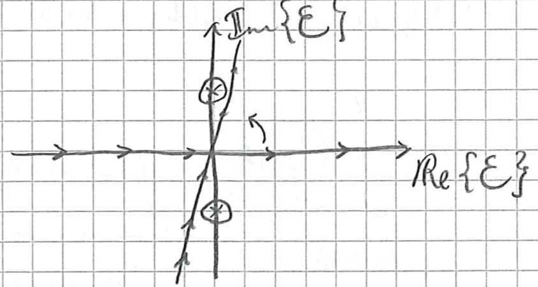
Claim: $W_E[F] = W_E[0]\, \exp\left(\frac{1}{2}\int F(\tau)D_E(\tau-\tau')F(\tau')\,d\tau\,d\tau'\right)$

Proof: As in part I.

No need for $\omega^2 = \lim\limits_{\varepsilon\to 0^+} \omega^2 - i\varepsilon$ now, bcs. exponent is always negative.

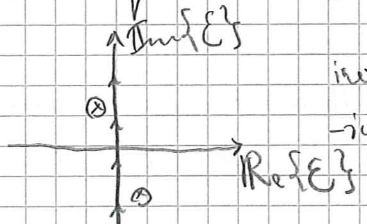
Now we have $D_E(\tau) := \int \frac{d\mathcal{E}}{2\pi} \frac{e^{-i\mathcal{E}\tau}}{\mathcal{E}^2+\omega^2}$

Poles at $\mathcal{E} = \pm i\omega$



Rotate int. path by $\pm\frac{\pi}{2} \mp i\varepsilon$ to get.

This is equiv. to rotation by $\frac{\pi}{2}$ and pushing the poles:



$i\omega \mapsto i\omega - \varepsilon$ $\qquad \mathcal{E} := iE$

$-i\omega \mapsto -i\omega + \varepsilon$

For $\mathcal{E} := -iE$

$$\int\limits_{\mathbb{R}} \frac{d\mathcal{E}}{2\pi} \frac{e^{-i\mathcal{E}\tau}}{\mathcal{E}^2+\omega^2} = \int\limits_{+i\mathbb{R}} \frac{dE\,(-i)}{2\pi} \frac{e^{-i(-iE)\tau}}{-E^2+\omega^2+i\varepsilon} = +i\int\limits_{\mathbb{R}} \frac{dE}{2\pi} \frac{e^{-E\tau}}{E^2-\omega^2-i\varepsilon}$$

In the int. of the action, $\tau := it$

$$-\int\limits_{\mathbb{R}} [(T(q))(\tau) + (V(q))(\tau)]\,d\tau \overset{\tau := it}{=} -\int\limits_{\mathbb{R}} [-T(q)(t) + V(q)(t)]\, i\,dt$$

$$= +i\int\limits_{\mathbb{R}} [T(q)(t) - V(q)(t)]\,dt$$

Thus we find

$$\boxed{\mathcal{D}(t) = \mathcal{D}_E(it)}$$

The generating functional (Peskin pp. 289)

Why did we add driving to the system?

Usually we want to compute quantities of the form:

$\langle e^{-iHt_f}\delta_{x_f},\ T(\hat{X}(t_1)\cdots\hat{x}(t_n))\, e^{-iHt_i}\delta_{x_i}\rangle$ t_j given times.

where $\hat{x}(t_j)$ is the Heisenberg-picture position op. at time t_j:

$$\hat{x}(t_j) \equiv e^{-iHt_j}\,\hat{x}\,e^{+iHt_j}$$

and T is the time-ordering operator.

For brevity also denote $e^{-iHt}\delta_x$ by $\delta_{x,t}$.

By discretizing we can show that (Peskin pp. 284)

$$\langle \delta_{x_f,t_f},\ T(\hat{x}(t_1)\cdots\hat{x}(t_n))\,\delta_{x_i,t_i}\rangle =$$

$$= \int_{\{q:\mathbb{R}\to\mathbb{R}:\ q(t_i)=x_i,\ q(t_f)=x_f\}} \mathcal{D}q\ \underbrace{q(t_1)\cdots q(t_n)}_{\text{numbers which depend on the path } q}\ \exp(iS(q))$$

But observe that if the Hamiltonian contain a source term (as ours does) then taking functional derivatives w.r.t. F we get:

$$\frac{\delta}{\delta F(t_j)}\exp(iS(q)) = \exp(iS(q))\,\frac{\delta}{\delta F(t_j)}\,iS(q)$$

$$= e^{iS}\, i\int_{t_i}^{t_f}\left[\frac{\delta}{\delta F(t_j)}\,F(t)\,q(t)\right]dt$$

$$= e^{iS}\, i\int_{t_i}^{t_f}\delta(t-t_j)\,q(t)\,dt$$

$$= e^{iS}\, i q(t_j)$$

As a result we find:

$$\langle \delta_{x_f,t_f}, T(\hat{x}(t_1)\cdots\hat{x}(t_n))\,\delta_{x_i,t_i}\rangle =$$

$$= \left(\frac{1}{i}\frac{\delta}{\delta F(t_1)}\cdots\frac{1}{i}\frac{\delta}{\delta F(t_n)}\langle \delta_{x_f,t_f}, \delta_{x_i,t_i}\rangle\right)\Bigg|_{F=0}$$

But we have just computed $\langle \delta_{x_f,t_f}, \delta_{x_i,t_i}\rangle$ for $t_i \to -\infty$, $t_f \to +\infty$

As a result we find:

$$\lim_t \langle \delta_{x_f,t_f}, T(\hat{x}(t_1)\cdots\hat{x}(t_n))\,\delta_{x_i,t_i}\rangle =$$

$$= \left(\lim_t \langle \delta_{x_f,t_f}, \delta_{x_i,t_i}\rangle\right)\Bigg|_{F=0} \left(\frac{1}{i}\frac{\delta}{\delta F(t_1)}\cdots\frac{1}{i}\frac{\delta}{\delta F(t_n)} e^{-\frac{i}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(t)D(t-s)F(s)\,dt\,ds}\right)\Bigg|_{F=0}$$

Example w/ $T(\hat{x}(t_1)\hat{x}(t_2))$:

$$\frac{1}{i}\frac{\delta}{\delta F(t_2)} e^{-\frac{i}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(t)D(t-s)F(s)\,dt\,ds} = \frac{1}{i} e^{\cdots}\left(-\frac{i}{2}\right)\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\delta(t-t_2)D(t-s)F(s) +$$

$$+ F(t)D(t-s)\delta(s-t_2)\Big]dt\,ds$$

$$= -\frac{1}{2}e^{\cdots}\left(\int_{\mathbb{R}} D(t_2-s)F(s)\,ds + \int_{\mathbb{R}} F(t)D(t-t_2)\,dt\right)$$

$$= -e^{\cdots}\int_{\mathbb{R}} D(t_2-t)F(t)\,dt$$

Another derivative gets us:

$$\frac{1}{i}\frac{\delta}{\delta F(t_1)}\left(-e^{\cdots}\int_{\mathbb{R}} D(t_2-t)F(t)\,dt\right) =$$

$$= -\frac{1}{i}\left(-\left(\int_{\mathbb{R}} D(t_2-t)F(t)\,dt\right)\left(-e^{\cdots}\int_{\mathbb{R}} D(t_1-t)F(t)\,dt\right) -\right.$$

$$\left. - e^{\cdots} D(t_2-t_1)\right)$$

When setting the source to zero we get simply $\boxed{D(t_2-t_1)}$.

This is also useful for perturbation theory.
We can show that if we add some perturbation L' to the Lagrangian: $L := L_0 + L'$

Then
$$\underbrace{W[F]}_{\text{full theory}} = \frac{\exp\left(i\int L_I\left(\frac{1}{i}\frac{\delta}{\delta F(t)}\right)dt\right) W_0[F]}{\left\{\left(\exp\left(i\int L_I\left(\frac{1}{i}\frac{\delta}{\delta F(t)}\right)dt\right) W_0[F]\right)\Big|_{F=0}\right\}}$$

So that we have a recipe to obtain arbitrary correlation functions of some perturbed theory.

<u>Example:</u> Perturbed Harmonic Oscillator

$$H(q) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 - \frac{g}{m!}q^m \qquad \text{for } m \geq 3.$$