Pricing American Options Under Variance Gamma*

Ali Hirsa† Dilip B. Madan‡

August 16, 2001

Abstract

We derive a form of the partial integro-differential equation (PIDE) for pricing American options under variance gamma (VG) process. We then develop a numerical algorithm to solve for values of American options under variance gamma model. In this study, we compare the exercise boundary and early exercise premia between geometric VG law and geometric Brownian motion (GBM). We find that GBM premia are understated and hence we conclude that further work is necessary in developing fast efficient algorithms for solving PIDE's with a view to calibrating stochastic processes to a surface of American option prices.

1 Introduction

A number of authors have recently proposed the use of infinite activity pure jump Lévy processes as the dynamics for the asset price (Eberlein, Keller and Prause [7], Barndorff-Nielsen [2] and Madan, Carr and Chang [11]). Further it is argued in Geman, Madan and Yor [8] that such processes are the norm when it is recognized that time changes with martingale components are involved in describing the price evolution. At an empirical level, Carr, Geman, Madan and Yor [5] recognize that the infinite activity of such Lévy processes effectively synthesizes the role of a diffusion component and reject the introduction of an additional diffusion component in the presence of a Lévy process that has a Lévy density integrating to infinity. Much of this research has focused attention on the time series or the prices of European options.

This paper develops efficient procedures for pricing American options when the underlying asset price has dynamics given by a pure jump infinity activity Lévy process. The methods are illustrated using the example of the variance gamma process introduced in Madan, Carr and Chang [11]. They are easily adapted to other processes in this class or jump diffusion models as proposed for example by Bates [3] or Duffie, Pan and Singleton [6].

We first derive a form of the partial integro-differential equation (PIDE) in the value function of the American claim particularly suited to our proposed numerical implementation. This result is followed by developing a numerical scheme which computes an approximate solution to the PIDE. At this stage we do not perform any error analysis of our scheme.

*Ali Hirsa would like to thank Peter Carr for his important suggestions, comments, and discussion in this paper. Errors are our own responsibility.

†Morgan Stanley ali.hirsa@morganstanley.com

‡Robert H. Smith School of Business, University of Maryland at College Park dbm@rhsmith.umd.edu

However, we claim that the scheme is consistent, stable, and converging. This claim is based on the observation that the numerical values for European options converge to the closed form solutions presented in Madan, Carr and Chang [11].

For the financial relevance of the derived American option prices we identify the risk neutral variance gamma process by calibration of S&P 500 market prices for European options. As a benchmark we employ a geometric Brownian motion taken at the implied volatility for the option. Results are presented on comparing the variance gamma and Black-Merton-Scholes early exercise premiums and optimal exercise boundaries.

An outline of the paper is as follows. The variance gamma (VG) process and VG European option pricing is summarized in §2. In §3, we derive the particular form of the partial integro-differential equation (PIDE) (suitable for our numerical implementation) in the value of an American option when the underlying asset price follows VG dynamics risk neutrally. This PIDE is discretized in §4. Numerical experiments and results are presented in §5. §6 concludes with a summary and comments on further research.

2 The Variance Gamma Model

The VG stock price process has no continuous martingale component. It is an example of a pure jump process having an infinite number of jumps in any interval of time. Most of the jumps are arbitrarily small in size with only finitely many jumps of any prespecified size. Other examples include the process of independent stable increments studied by McCulloch [12] as well as the references cited earlier. The process may be presented in a variety of ways and we describe it first as time changed Brownian motion with drift.

Let $b(t;\theta,\sigma) \equiv \theta t + \sigma W(t)$, be a Brownian motion with constant drift rate θ and volatility σ, where $W(t)$ is standard Brownian motion. Now define the gamma process $\gamma(t;\nu)$ with independent gamma increments over intervals of length h with mean h and variance rate νh.

The three parameter *VG process* $X(t;\sigma,\theta,\nu)$ is defined by

$$X(t;\sigma,\theta,\nu) = b(\gamma(t;1,\nu),\theta,\sigma).$$

We see that the process $X(t)$ is a Brownian motion with drift evaluated at a gamma time change.
The characteristic function for the time t level of the VG process is[1]

$$\phi_{X(t)}(u) = \mathbb{E}(e^{iuX(t)}) = \left(\frac{1}{1 - iu\theta\nu + \sigma^2 u^2 \nu/2}\right)^{\frac{t}{\nu}}. \tag{2.1}$$

This characteristic function is infinitely divisible with a zero continuous martingale component and a Lévy measure identified in the unique decomposition of the characteristic function in accordance with the Lévy Khintchine theorem by

$$F(dt,dx) = dt\frac{e^{\theta x/\sigma^2}}{\nu|x|} exp\left(-\frac{(2/\nu + \theta^2/\sigma^2)^{1/2}|x|}{\sigma}\right) dx. \tag{2.2}$$

[1]The VG process may be expressed as the difference of two independent increasing gamma processes and therefore the characteristic function of it is obtained as the product of characteristic function of the two gamma processes which is well known.

We note that for $\theta = 0$ the Lévy density is symmetric and the process has zero skewness. Negative values for θ result in negative skewness while the parameter ν builds kurtosis. For $\theta = 0$ the kurtosis is $3(1+\nu)$. The infinite activity of the process is observed on noting that the Lévy density has the behavior of $(1/|x|)$ in a neighborhood of zero and in particular that hence, the integral of the Lévy density is infinite.

2.1 The VG European Option Pricing Formulas

The VG dynamics of the stock price mirrors that of geometric Brownian motion for a stock paying a continuous divdend yield of q in an economy with a constant continuously compounded interest rate of r. The risk neutral drift rate for the stock price is $r - q$ and the forward stock price is modeled as the exponential of a VG process normalized by its expectation. Specifically, let $S(t)$ be the stock price at time t. The VG risk neutral process for the stock price is given by

$$S(t) = S(0)e^{(r-q)t+X(t)+\omega t}, \tag{2.3}$$

where the normalization factor $e^{\omega t}$ ensuring that $E[S(t)] = S(0)e^{(r-q)t}$, requires that

$$\omega = \frac{1}{\nu}\ln(1 - \sigma^2\nu/2 - \theta\nu).$$

By definition of risk neutrality, the price of a European call option with strike K and maturity T is

$$c(S(0); K, t) = e^{-rt}\mathbb{E}_t((S(t) - K)^+).$$

Theorem 2.1 *The European call option price on a stock, when the risk neutral dynamics of the stock price is given by the VG process (2.3) for risk neutral parameters σ, ν, θ is*

$$\begin{aligned} c(S(0); K, T) &= S(0)e^{-qT}\Psi(d\sqrt{(1-c_1)/\nu}, (\alpha+\sigma)\sqrt{\nu/(1-c_1)}, T/\nu) \\ &\quad -Ke^{-rT}\Psi(d\sqrt{(1-c_2)/\nu}, \alpha\sqrt{\nu/(1-c_2)}, T/\nu), \end{aligned}$$

where

$$d = \frac{1}{s}\left[\ln(S(0)/K) + (r-q)T + \frac{T}{\nu}\left(\frac{1-\nu(\alpha+s)^2/2}{1-\nu\alpha^2/2}\right)\right],$$

$$\alpha = -\frac{\theta}{\sigma^2},$$

$$s = \frac{\sigma}{\sqrt{1+(\theta/\sigma)^2(\nu/2)}},$$

$c_1 = \nu(\alpha+s)^2$, $c_2 = \nu\alpha^2/2$, *and the function Ψ is defined in terms of the modified Bessel function of second kind and the degenerate hyper-geometric function of two variables.*

Proof. See [11] Q.E.D

By put-call parity the VG European put option price is seen to be

$$\begin{aligned} p(S(0); K, T) &= Ke^{-rT}\Psi(-d\sqrt{(1-c_2)/\nu}, \alpha\sqrt{\nu/(1-c_2)}, T/\nu)) \\ &\quad -S(0)e^{-qT}\Psi(-d\sqrt{(1-c_1)/\nu}, \alpha\sqrt{\nu/(1-c_1)}, T/\nu)). \end{aligned}$$

3 VG American option pricing

We shall consider the price of an American put option of strike K and maturity T when the risk neutral dynamics for the stock price are given by equation (2.3). By the Markov property for the underlying dynamics this price is given by a function $V(S(t), t\,; K, T)$. This price is defined as

$$V(S(t), t; K, T) = \sup_{t \leq \tau \leq T} e^{-r\tau} E\left((K - S(\tau))^+\right),$$

where the sup is taken over all stopping times τ defined on the probability space with respect to the filtration generated by the stock price. It may be shown (include reference) that for each t there exists a critical stock price $S^*(t)$ such that if $S(t) \leq S^*(t)$ the value of the American put option is the value of immediate exercise or $K - S(t)$ while for $S(t) > S^*(t)$ the value exceeds this immediate exercise value. The curve $S^*(t)$ viewed as a function of time is referred to as the critical exercise boundary while the region

$$\mathcal{C} = \{(S, t)\,|S > S^*(t)\}$$

is called the continuation region. The complement $\mathcal{E}$ of the continuation region is the exercise region.

The value of the American put in the exercise region is known and it only remains to determine the value in the continuation region. It may further be shown that the discounted price of the option is a martingale in the continuation region and hence the infinitessimal generator $\mathcal{L}$, of the underlying Markov process applied to this discounted price,

$$\mathcal{L}\left(e^{-rt}V(S(t), t\right) = 0. \tag{3.4}$$

We develop the specific form of the PIDE that we solve numerically as an application of these results.

3.1 Derivation of the PIDE

The underlying Markov process to which the stock price is adapted, we supoose to be given by $Y(t) = \log(S(t))$. It follows from equation (2.3) that

$$Y(t) = Y(0) + (r - q + \omega)t + X(t; \sigma, \nu, \theta).$$

Noting that $X(t; \sigma, \nu, \theta)$ is a pure jump process that may be written as the sum of all its jumps we may write the explicit semimartingale decomposition for $Y(t)$ as follows. Let μ denote the integer valued random measure associated with the jumps of the process $X(t; \sigma, \nu, \theta)$ and let ν be its predictable compensator. For the specific case of the VG process we have that

$$\begin{aligned}
\nu(dx, dt) &= k(x)dxdt \\
k(x) &= \frac{e^{-\lambda_p x}}{\nu x}\mathbf{1}_{x>0} + \frac{e^{-\lambda_n|x|}}{\nu|x|}\mathbf{1}_{x<0} \\
\lambda_p &= \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} - \frac{\theta}{\sigma^2} \\
\lambda_n &= \left(\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2\nu}\right)^{\frac{1}{2}} + \frac{\theta}{\sigma^2}
\end{aligned}$$

The semimartingale decomposition for $Y(t)$ is given by

$$Y(t) = Y(0) + (r - q + \omega)t + x * \nu_t + x * (\mu - \nu)_t$$

where the operation $*$ signifies a double integration with respect to x, t in the domain $\mathbb{R} \times [0, t]$, and $\mathbb{R}$ denotes the real line.

The infinitessimal generator for the Markov process $Y(t)$ applies to functions $f(Y, t)$ and is given by

$$\begin{aligned} \mathcal{L}f &= \left[\frac{\partial}{\partial t} + (r - q + \omega + \int_{-\infty}^{\infty} xk(x)dx)\frac{\partial}{\partial Y}\right] f + \\ & \int_{-\infty}^{\infty} \left[f(Y + x, t) - f(Y, t) - x\frac{\partial}{\partial Y} f\right] k(x)dx \end{aligned} \tag{3.5}$$

Regarding the value function in the continuation region as a function of the Markov process $Y(t)$ and applying the operator (3.5) to the discounted value function as stated in equation (3.4) yields the PIDE

$$\begin{aligned} \int_{-\infty}^{+\infty} \left[V(S_{t-}e^x, t) - V(S_{t-}, t) - \frac{\partial V}{\partial S}(S_{t-}, t)S_{t-}(e^x - 1)\right] k(x)dx & \\ + \frac{\partial V}{\partial t}(S_t, t) + (r - q)S_t\frac{\partial V}{\partial S}(S_t, t) - rV(S_t, t) &= 0 \end{aligned} \tag{3.6}$$

By making the change of variables $x = \ln S$ and $\tau = T - t$ we obtain, noting

$$\begin{aligned} w(x, \tau) &= V(S, t), \\ \frac{\partial w}{\partial x}(x, \tau) &= S\frac{\partial V}{\partial S}(S, t), \\ \frac{\partial w}{\partial \tau}(x, \tau) &= -\frac{\partial V}{\partial t}(S, t), \\ w(x + y, \tau) &= V(Se^y, t), \end{aligned}$$

the following PIDE in the function $w(x, \tau)$,

$$\begin{aligned} -\frac{\partial w}{\partial \tau}(x, \tau) + (r - q)\frac{\partial w}{\partial x}(x, \tau) - rw(x, \tau) & \\ + \int_{-\infty}^{+\infty} \left[w(x + y, \tau) - w(x, \tau) - \frac{\partial w}{\partial x}(x, \tau)(e^y - 1)\right] k(dy) &= 0 \end{aligned}$$

This equation may be equivalently written as [2]

$$-\frac{\partial w}{\partial \tau}(x, \tau) + (r - q + \omega)\frac{\partial w}{\partial x}(x, \tau) - rw(x, \tau)$$

[2] By a straightforward algebraic calculations, we can show

$$\int_{-\infty}^{+\infty} (e^y - 1)k(dy) = -\omega$$

Alternatively, by definition from Lévy we have

$$k(dy) = \lim_{t \to 0} \frac{P(y_t \in dy|\, y_0 = 0)}{t}$$

and

$$\mathbb{E}(e^{X_t}) = e^{-\omega t}$$

$$+\int_{-\infty}^{+\infty}\left[w(x+y,\tau)-w(x,\tau)\right]k(y)dy \quad = \quad 0 \tag{3.7}$$

We extend the $PIDE$ to the entire region as in the exercise region we know the value function $w(x,t)=(K-e^x)^+$ and applying the infinitessimal generator to this known value function in the exercise region yields the equation $\mathcal{L}w=\delta(x)$ where in particular we write that

$$\begin{aligned}&-\frac{\partial w}{\partial \tau}(x,\tau)+(r-q+\omega)\frac{\partial w}{\partial x}(x,\tau)-rw(x,\tau)\\&+\int_{-\infty}^{+\infty}\left[w(x+y,\tau)-w(x,\tau)\right]k_{\mathbb{Q}}(y)dy\\=\;&\delta(x).\end{aligned}$$

The function $\delta(x)$ is a often called the *dividend process*. This is best seen using the fact that $w(x,\tau)=K-e^x$ for $x\leq x(\tau)$ and hence in this region we get

$$\delta(x)=-rK+qe^x+\int_{x(\tau)-x}^{\infty}\left[w(x+y,\tau)-(K-e^{x+y})\right]k_{\mathbb{Q}}(y)dy \tag{3.8}$$

and consistent with the demonstration by Carr, Jarrow and Myneni [4], one must extract from the American option holder the interest on the strike less the dividend yield for the time the stock spends in the exercise region to get the value back to that of a European option. For a jump process like the variance gamma this amount is further reduced by the expected shortfall the stop loss start gain strategy may experience on account of jumping back into the continuation region as explained further in Gukhal [9].

Substituting the dividend definition of equation (3.8) back into the PIDE we obtain the PIDE in $w(x,t)$ over the entire region as

$$\begin{aligned}&\frac{\partial w}{\partial \tau}(x,\tau)-(r-q+\omega)\frac{\partial w}{\partial x}(x,\tau)+rw(x,\tau)\\&-\int_{-\infty}^{+\infty}\left[w(x+y,\tau)-w(x,\tau)\right]k_{\mathbb{Q}}(y)dy\\-\mathbf{1}_{x<x(\tau)}&\left\{rK-qe^x-\int_{x(\tau)-x}^{\infty}\left[w(x+y,\tau)-(K-e^{x+y})\right]k_{\mathbb{Q}}(y)dy\right\}=0.\end{aligned} \tag{3.9}$$

This PIDE must be solved subject to initial condition

$$w(x,0)=(K-e^x)^+, \tag{3.10}$$

and boundary conditions

$$\frac{\partial^2 w}{\partial x^2}(-\infty,\tau)-\frac{\partial w}{\partial x}(-\infty,\tau) \quad = \quad 0 \quad \forall\tau, \tag{3.11}$$

$$\frac{\partial^2 w}{\partial x^2}(+\infty,\tau)-\frac{\partial w}{\partial x}(+\infty,\tau) \quad = \quad 0 \quad \forall\tau. \tag{3.12}$$

Thus, we can write

$$\frac{1}{t}\int_{-\infty}^{+\infty}(e^y-1)\;P(y_t\in dy|\;y_0=0)dy=\frac{e^{-\omega t}-1}{t}$$

by taking the limit as t approaches zero we get

$$\int_{-\infty}^{+\infty}(e^y-1)k(y)dy=-\omega$$

4 Discretization

In our finite difference discretization, we use a mix approach. On the evaluation of the jump integral, we have an explicit approach and on the PDE fully implicit approach. The rationale behind this scheme is that it would be computationally less expensive. For an American put with maturity T, we consider M equally sub-intervals in τ-direction. For x-direction we assume N equal sub-intervals on $[x_{\min}, x_{\max}]$. Thus, we have the following mesh on $[x_{\min}, x_{\max}] \times [0, T]$

$$\begin{aligned} D = \big\{ &(x_i, \tau_j) \in \mathbb{R}^+ \times \mathbb{R}^+ \,| x_i = i\Delta x, : i = 0, 1, \ldots, N, \\ &\tau_j = j\Delta\tau, : j = 0, 1, \ldots, M,\ \Delta x = (x_{\min} - x_{\max})/N, : \Delta\tau = T/M \,\big\} \end{aligned}$$

Let $w_{i,j}$ be the discrete values of $w(x_i, \tau_j)$ on D. Using first order finite difference approximation for $\frac{\partial w}{\partial \tau}$ and central difference for $\frac{\partial w}{\partial x}$ we obtain the following discrete equation at point (x_i, τ_{j+1})

$$\begin{aligned} & \frac{1}{\Delta\tau}\left(w_{i,j+1} - w_{i,j}\right) - (r - q + \omega)\frac{1}{2\Delta x}(w_{i+1,j+1} - w_{i-1,j+1}) + r w_{i,j+1} \\ & - \int_{-\infty}^{+\infty} \left(w(x_i + y, \tau_j) - w(x_i, \tau_j)\right) k(y) dy \\ & - \mathbf{1}_{x_i < x(\tau_j)} \left\{ rK - q e^{x_i} - \int_{x(\tau_j) - x_i}^{\infty} \left[u(x_i + y, \tau_j) - (K - e^{x_i + y}) \right] k(y) dy \right\} \\ = \;& 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} & (r - q + \omega)\frac{\Delta\tau}{2\Delta x} w_{i-1,j+1} + (1 + r\Delta\tau) w_{i,j+1} - (r - q + \omega)\frac{\Delta\tau}{2\Delta x} w_{i+1,j+1} \\ = \;& w_{i,j} + \Delta\tau \int_{-\infty}^{+\infty} \left(w(x_i + y, \tau_j) - w(x_i, \tau_j)\right) k(y) dy \\ & + \Delta\tau \; \mathbf{1}_{x_i < x(\tau_j)} \left\{ rK - q e^{x_i} - \int_{x(\tau_j) - x_i}^{\infty} \left[w(x_i + y, \tau_j) - (K - e^{x_i + y}) \right] k(y) dy \right\} \end{aligned}$$

where $w_{i,0} = (K - e^{x_i})^+$, $x(\tau_0) = K$ and

$$x(\tau_j) = \min_{x_i} \{ x_i : w(x_i, \tau_j) - (K - e^{x_i})^+ > 0 \} \text{ for } j = 1, \ldots, M.$$

4.1 Evaluation of the Jump Integral

There are various approaches on evaluating the jump integral. In [10], *not-a-knot* cubic spline [1] interpolation was applied at the known discrete values $w_{i,j}$ and then the interpolant and the Lévy density were used to evaluate the integral by means of adaptive recursive Newton-Cotes 8 panel rule. Tavella and Randall [13] use Romberg integration for evaluating the integral.

In the process of evaluation the jump integral, we divide it to six sub-intervals. The Lévy measure $k(y)$ is singular at $y = 0$ but that would not cause any problem in the process of approximation of the integral as we would see later.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy &= \int_{-\infty}^{x_0-x_i} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \\
&+ \int_{x_0-x_i}^{-\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \\
&+ \int_{-\Delta x}^{0} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \\
&+ \int_{0}^{+\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \\
&+ \int_{+\Delta x}^{x_N-x_i} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \\
&+ \int_{x_N-x_i}^{+\infty} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy
\end{aligned}$$

For $y \in [-\Delta x, 0]$ and assuming that Δx is small enough, one can write

$$w(x_i+y,\tau_j) - w_{i,j} = \frac{w_{i-1,j} - w_{i,j}}{\Delta x} y + O(y^2).$$

Similarly, for $y \in [0, \Delta x]$ we have

$$w(x_i+y,\tau_j) - w_{i,j} = \frac{w_{i+1,j} - w_{i,j}}{\Delta x} y + O(y^2).$$

Therefore

$$\int_{-\Delta x}^{0} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \cong \frac{1}{\nu \Delta x \lambda_n}(1 - e^{-\lambda_n \Delta x})(w_{i-1,j} - w_{i,j}),$$

and

$$\int_{0}^{+\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy \cong \frac{1}{\nu \Delta x \lambda_p}(1 - e^{-\lambda_p \Delta x})(w_{i+1,j} - w_{i,j}).$$

For $y \in (x_0 - x_i, -\Delta x)$, we do the following

$$\int_{x_0-x_i}^{-\Delta x} \left(w(x_i+y,\tau_j) - w_{i,j}\right) k(y)dy = \sum_{k=1}^{i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w(x_i-y,\tau_j) - w_{i,j}\right) \frac{e^{-\lambda_n} y}{\nu y} dy$$

Using linear interpolation on interval $y \in [k\Delta x, (k+1)\Delta x]$, we can write $w(x_i - y, \tau_j)$ as follows

$$w(x_i - y, \tau_j) \cong w_{i-k,j} + \frac{w_{i-k-1,j} - w_{i-k,j}}{\Delta x}(y - k\Delta x),$$

and therefore we obtain the following[3]

$$= \sum_{k=1}^{i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w_{i-k,j} + \frac{w_{i-k-1,j} - w_{i-k,j}}{\Delta x}(y - k\Delta x) - w_{i,j} \right) \frac{e^{-\lambda_n} y}{\nu y} dy$$

[3] $\mathrm{expint}(\xi) = \int_\xi^\infty \frac{e^{-y}}{y} dy$

$$
\begin{aligned}
&= \sum_{k=1}^{i-1} \frac{1}{\nu}\left(w_{i-k,j} - w_{i,j} - k(w_{i-k-1,j} - w_{i-k,j})\right)\left\{\text{expint}(k\Delta x\lambda_n) - \text{expint}((k+1)\Delta x\lambda_n)\right\} \\
&+ \sum_{k=1}^{i-1} \frac{w_{i-k-1,j} - w_{i-k,j}}{\lambda_n \nu \Delta x}\left(e^{-\lambda_n k\Delta x} - e^{-\lambda_n (k+1)\Delta x}\right)
\end{aligned}
$$

Similarly, for $y \in [k\Delta x, (k+1)\Delta x]$, a linear approximation yields

$$
w(x_i + y, \tau_j) \cong w_{i+k,j} + \frac{w_{i+k+1,j} - w_{i+k,j}}{\Delta x}(y - k\Delta x),
$$

and we get for $y \in (\Delta x, x_N - x_i)$

$$
\int_{\Delta x}^{x_N - x_i} \left(w(x_i + y, \tau_j) - w_{i,j}\right) k_{\mathbb{Q}}(y)dy = \sum_{k=1}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w(x_i + y, \tau_j) - w_{i,j}\right) \frac{e^{-\lambda_p y}}{\nu y} dy
$$

$$
\begin{aligned}
&= \sum_{k=1}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w_{i+k,j} + \frac{w_{i+k+1,j} - w_{i+k,j}}{\Delta x}(y - k\Delta x) - w_{i,j}\right) \frac{e^{-\lambda_p y}}{\nu y} dy \\
&= \sum_{k=1}^{N-i-1} \frac{1}{\nu}\left(w_{i+k,j} - w_{i,j} - k(w_{i+k+1,j} - w_{i+k,j})\right)\left\{\text{expint}(k\Delta x\lambda_p) - \text{expint}((k+1)\Delta x\lambda_p)\right\} \\
&+ \sum_{k=1}^{N-i-1} \frac{w_{i+k+1,j} - w_{i+k,j}}{\lambda_p \nu \Delta x}\left(e^{-\lambda_p k\Delta x} - e^{-\lambda_p (k+1)\Delta x}\right)
\end{aligned}
$$

For $y \in (-\infty, x_0 - x_i)$

$$
\int_{-\infty}^{x_0 - x_i} \left(w(x_i + y, \tau_j) - w_{i,j}\right) k_{\mathbb{Q}}(y)dy = \int_{i\Delta x}^{\infty} \left(w(x_i - y, \tau_j) - w_{i,j}\right) \frac{e^{-\lambda_n y}}{\nu y} dy
$$

We assume that $w(x_i - y, \tau_j) = K - e^{x_i - y}$[4] in that interval. Hence

$$
\begin{aligned}
&= \int_{i\Delta x}^{\infty} \left(K - e^{x_i - y} - w_{i,j}\right) \frac{e^{-\lambda_n y}}{\nu y} dy \\
&= \frac{1}{\nu}(K - w_{i,j})\text{expint}(i\Delta x\lambda_n) - \frac{1}{\nu} e^{x_i}\text{expint}(i\Delta x(\lambda_n + 1))
\end{aligned}
$$

For $y \in [(N-i)\Delta x, \infty)$, we assume that $w(x_i + y, \tau_j) = 0$.[5] Thus

$$
\begin{aligned}
\int_{x_N - x_i}^{+\infty} \left(w(x_i + y, \tau_j) - w_{i,j}\right) k_{\mathbb{Q}}(y)dy &= \int_{(N-i)\Delta x}^{\infty} \left(w(x_i + y, \tau_j) - w_{i,j}\right) \frac{e^{-\lambda_p y}}{\nu y} dy \\
&= -\frac{1}{\nu} w_{i,j}\text{expint}((N-i)\Delta x\lambda_p)
\end{aligned}
$$

[4] We choose x_0 small enough in such that $w(x_0, \tau_j) = K - e^{x_0}$ for all j. Thus it would be true for $w(x_i - y, \tau_j) = K - e^{x_i - y}$ as long as $x_i - y < x_0$. In the case of European, we assume that $w(x_i - y, \tau_j) = Ke^{-r\tau_j} - e^{x_i - y}e^{-q\tau_j}$.

[5] As explained before x_N is selected such that $w(x_N, \tau_j) = 0$. Therefore the assumption $w(x_i + y, \tau_j) = 0$ is valid as long as $x_i + y > x_N$.

4.2 Heaviside Term

The integral inside Heaviside term would be treated in the same manner

$$\int_{x(\tau_j)-x_i}^{\infty} \left[w(x_i+y,\tau_j)-(K-e^{x_i+y})\right] k(y)dy$$

$$\begin{aligned}
&= \sum_{k=l-i}^{N-i-1} \int_{k\Delta x}^{(k+1)\Delta x} \left(w_{i+k,j} + \frac{w_{i+k+1,j}-w_{i+k,j}}{\Delta x}(y-k\Delta x)\right) \frac{e^{-\lambda_p}y}{\nu y} dy \\
&- \frac{1}{\nu}\left\{K\mathrm{expint}((l-i)\Delta x\lambda_p) - e^{x_i}\mathrm{expint}((l-i)\Delta x(\lambda_p-1))\right\} \\
&= \sum_{k=l-i}^{N-i-1} \frac{1}{\nu}\left(w_{i+k,j}-w_{i,j}-k(w_{i+k+1,j}-w_{i+k,j})\right)\left(\mathrm{expint}(k\Delta x\lambda_p)-\mathrm{expint}((k+1)\Delta x\lambda_p)\right) \\
&+ \sum_{k=l-i}^{N-i-1} \frac{w_{i+k+1,j}-w_{i+k,j}}{\lambda_p\nu\Delta x}\left(e^{-\lambda_p k\Delta x}-e^{-\lambda_p(k+1)\Delta x}\right) \\
&- \frac{1}{\nu}\left\{K\mathrm{expint}((l-i)\Delta x\lambda_p) - e^{x_i}\mathrm{expint}((l-i)\Delta x(\lambda_p-1))\right\}
\end{aligned}$$

4.3 Difference Equation

After substitution, we obtain the following difference equation[6] at point (x_i,τ_{j+1})

$$Aw_{i-1,j+1}+Bw_{i,j+1}-Cw_{i+1,j+1} = w_{i,j}+\frac{\Delta\tau}{\nu}R_{i,j}+\Delta\tau\mathbf{1}_{x_i<x(\tau_j)}H_{i,j} \tag{4.13}$$

where

$$\begin{aligned}
A &= a-B_n \\
B &= 1+r\Delta\tau+B_n+B_p+\mathrm{expint}(i\Delta x\lambda_n)+\mathrm{expint}((N-i)\Delta x\lambda_p) \\
C &= a+B_p
\end{aligned}$$

$$\begin{aligned}
R_{i,j} &= \sum_{k=1}^{i-1}\left(w_{i-k,j}-w_{i,j}-k(w_{i-k-1,j}-w_{i-k,j})\right)\left\{\mathrm{expint}(k\Delta x\lambda_n)-\mathrm{expint}((k+1)\Delta x\lambda_n)\right\} \\
&+ \sum_{k=1}^{i-1}\frac{w_{i-k-1,j}-w_{i-k,j}}{\lambda_n\Delta x}\left(e^{-\lambda_n k\Delta x}-e^{-\lambda_n(k+1)\Delta x}\right) \\
&+ \sum_{k=1}^{N-i-1}\frac{1}{\nu}\left(w_{i+k,j}-w_{i,j}-k(w_{i+k+1,j}-w_{i+k,j})\right)\left\{\mathrm{expint}(k\Delta x\lambda_p)-\mathrm{expint}((k+1)\Delta x\lambda_p)\right\} \\
&+ \sum_{k=1}^{N-i-1}\frac{w_{i+k+1,j}-w_{i+k,j}}{\lambda_p\nu\Delta x}\left(e^{-\lambda_p k\Delta x}-e^{-\lambda_p(k+1)\Delta x}\right) \\
&+ K\mathrm{expint}(i\Delta x\lambda_n)-e^{x_i}\mathrm{expint}(i\Delta x(\lambda_n+1))
\end{aligned}$$

[6]In the case of $i=1$ or $i=N$ we impose boundary conditions.

T	S_0	r	q	σ	ν	θ
0.13972	1369.41	0.0533	0.011	0.17875	0.13317	-0.30649
0.21643	1369.41	0.0536	0.012	0.18500	0.22460	-0.28837
0.46575	1369.41	0.0549	0.011	0.19071	0.49083	-0.28113
0.56164	1369.41	0.0541	0.012	0.20722	0.50215	-0.22898

Table 1: Calibrated VG parameters for S&P 500 June 30, 1999

$$
\begin{aligned}
H_{i,j} &= rK - qe^{x_i} \\
&- \sum_{k=l-i}^{N-i-1} \frac{1}{\nu} \left(w_{i+k,j} - w_{i,j} - k(w_{i+k+1,j} - w_{i+k,j})\right) \left(\operatorname{expint}(k\Delta x \lambda_p) - \operatorname{expint}((k+1)\Delta x \lambda_p)\right) \\
&- \sum_{k=l-i}^{N-i-1} \frac{w_{i+k+1,j} - w_{i+k,j}}{\lambda_p \nu \Delta x} \left(e^{-\lambda_p k \Delta x} - e^{-\lambda_p (k+1)\Delta x}\right) \\
&+ \frac{1}{\nu} \left\{K \operatorname{expint}((l-i)\Delta x \lambda_p) - e^{x_i} \operatorname{expint}((l-i)\Delta x(\lambda_p - 1))\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
a &= (r - q - \omega) \frac{\Delta \tau}{2\Delta x} \\
B_n &= \frac{\Delta \tau}{\nu \Delta x \lambda n} \left(1 - e^{-\lambda_n \Delta x}\right) \\
B_p &= \frac{\Delta \tau}{\nu \Delta x \lambda p} \left(1 - e^{-\lambda_p \Delta x}\right)
\end{aligned}
$$

Assuming that at time τ_j value of $w_{i,j}$ are known, we solve a linear system to find $w_{i.j+1}$ for all i. Notice that in this scheme, we store the following four vectors (pre-calculated):

- $\operatorname{expint}(k\Delta x \lambda_n)$ for $k = 1, \ldots, N$,
- $\operatorname{expint}(k\Delta x \lambda_p)$ for $k = 1, \ldots, N$,
- $e^{-\lambda_n k \Delta x}$ for $k = 1, \ldots, N$,
- $e^{-\lambda_p k \Delta x}$ for $k = 1, \ldots, N$.

5 Numerical experiments

We separately calibrated VG parameters for each maturity. Using out-of-the-money call and put European option prices for S&P 500 June 30, 1999, we obtained the following parameters

As we see in Table 1 as maturity gets larger the annualized kurtosis parameter ν increases and annualized skewness θ decreases. The increase in ν is slower than the increase in maturity and this is consistent with an approach to normality, though at a rate slower than would be the case if ν were constant or falling. The decrease in θ is also broadly consistent with the apporach to normality. Table 2 contains the Black-Scholes implied volatility for

Strike	$T = 0.13972$	$T = 0.21643$	$T = 0.46575$	$T = 0.56164$
1200	0.2675	0.2737	0.2801	0.2868
1220	0.2592	0.2662	0.2743	0.2818
1240	0.2508	0.2587	0.2686	0.2768
1260	0.2422	0.2509	0.2629	0.2718
1280	0.2334	0.2431	0.2571	0.2667
1300	0.2244	0.2351	0.2513	0.2616
1320	0.2152	0.227	0.2455	0.2565
1340	0.2057	0.2187	0.2396	0.2514
1360	0.1961	0.2102	0.2337	0.2462
1380	0.1863	0.2016	0.2277	0.2409
1400	0.1767	0.193	0.2217	0.2357

Table 2: Implied volatility for S&P 500 options on June 30, 1999

these option prices. We observe a significant skewness in these implied volatilities with a drop of 10 volatility points over the specified strike range. Table 3 contains the early exercise premiums from pricing American options under the variance gamma and geometric Brownian motion dynamics respectively. The American pricing under geometric Brownian motion is conducted at the implied volatility for the option reported in Table 2. We observe that across all strikes and maturities, the VG early exercise premiums dominate those from geometric Brownian motion. This suggests that the traditional practice of adding geometric Brownian motion based American option premia consistent with the implied volatility of a European price quote to infer an American option price is biased downward with respect to the American option price consistent with the underlying VG dynamics for the stock price. The differences can be substantial and for example for the 1320 strike with maturity .2164 it is about a 1/3 of the American option value under geometric Brownian motion. We are led to conclude that though the pricing of American options under the right underlying dynamics may be difficult, it is important from the perspective of correctly accounting for the values of these instruments. We also compare the exercise boundary for VG and GBM for strike $K = 1300$ and maturity $T = 0.56164$ in Figure 1. This example exhibits a smaller continuation region we may be associated with an earlier exercise. The exact timing is difficult to comment on as the differences in the underlying dynamics enter into the issues of passage times to these boundaries.

6 Conclusion and Future Work

We derived a PIDE for the VG American option and priced its values numerically. As a consistency check, we checked the European prices using the algorithm and solutions and compared these with results from closed form formulas for European options.

The pricing of American options for Lévy processes generally will involve the numerical solution of partial integro-differential equations that are similar to the one illustrated here for the case of the variance gamma process. The methods developed here would therefore be applicable to a wide class of problems. In particular our combination of analytical and numerical approaches to the singularity at zero for infinite activity Lévy processes should prove useful in many contexts.

Maturity	$T_1 = 0.13972$		$T_2 = 0.21643$		$T_3 = 0.46575$		$T_4 = 0.56164$	
Strike	GBM	VG	GBM	VG	GBM	VG	GBM	VG
1200	0.052	0.025	0.036	0.063	0.443	0.539	0.828	1.041
1220	0.026	0.070	0.123	0.124	0.647	0.677	0.710	1.250
1240	0.010	0.117	0.165	0.191	0.569	0.835	1.131	1.489
1260	0.065	0.159	0.081	0.253	0.918	0.997	1.192	1.742
1280	0.113	0.196	0.251	0.322	0.856	1.185	1.483	2.023
1300	0.164	0.239	0.359	0.392	1.298	1.388	1.756	2.339
1320	0.231	0.428	0.415	0.602	1.318	1.681	1.966	2.741
1340	0.328	0.608	0.465	0.791	1.856	2.008	2.462	3.178
1360	0.462	0.869	0.745	1.078	1.999	2.392	2.686	3.687
1380	0.630	1.103	1.069	1.374	2.664	2.795	3.400	4.241
1400	1.066	1.483	1.475	1.678	3.112	3.291	3.771	4.861

Table 3: Early exercise premiums for variance gamma (VG) and geometric Brownian motion (GBM)

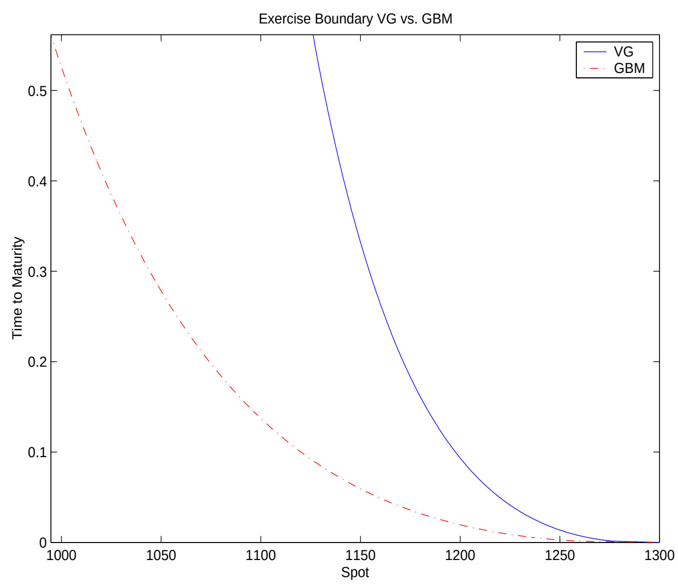

Figure 1: Comparsion of exercise boundaries between variance gamma (VG) and geometric Brownian motion (GBM) for strike $K = 1300$ and maturity $T = 0.56164$.

A comparison of early exercise premia for the variance gamma process with geometric Brownian motion, with the latter computed at the European implied volatility of the option revealed that the premia from early exercise is higher for the Lévy process. This calls into question the practice of deriving American option prices by adding Black-Scholes early exercise premia to European prices and suggests that we need to make progress on the efficient solution of partial integro-differential equations for a wide context of cases. These remarks apply as well to other exotic options like the array of barrier options. We welcome developments on these numerical issues.

References

[1] Kendall E. Atkinson. *An Introduction to Numerical Analysis*. John Wiley & Sons, second edition, 1989.

[2] Barndorff-Nielsen and N. Shepard. Non-Gaussian OU based models and some of their uses in financial economics. working paper no. 37, Center for Analytical Finance, University of Aarhus, 1999.

[3] David S. Bates. Post-'87 crash fears in S&P 500 options. *Journal of Econometrics*, 94:181–238, 2000.

[4] P. Carr, R.A. Jarrow, and R. Myneni. Alternative characterization of American put options. *Mathematical Finance*, (2):87–106, 1992.

[5] Peter Carr, Hélyette Geman, Dilip B. Madan, and Marc Yor. Stochastic Volatility for Lévy Processes. Feb 2001.

[6] D. Duffie, D. J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump diffusions. *Econometrica*, 68:1343–1376, 2000.

[7] E. Eberlein, U. Keller, and K. Prause. New insights into smile, mispricing and value at risk. *Journal of Business*, (71):371–406, 1998.

[8] H. Geman, D. B. Madan, and Marc Yor. Time changes for lévy processes. *Mathematical Finance*, 11:79–96, 2001.

[9] C.R. Gukhal. Analytical valuation of American options on jump-diffusion processes. *Mathematical Finance*, (11):97–115, 2001.

[10] Ali Hirsa. *Numerical Algorithms for Varaince Gamma and Convection Diffusion Equation*. PhD thesis, University of Maryland at College Park, Dec 1997.

[11] Dilip B. Madan, Peter Carr, and Eric C. Chang. The variance gamma process and option pricing. *European Finance Review*, 2:79–105, 1998.

[12] J. H. McCulloch. Continuous time processes with stable increments. *Journal of Business*, 51:601–619, October 1978.

[13] Domingo Tavella and Curt Randall. *Pricing Financial Instruments, The Finite Difference Method*. John Wiley & Sons, first edition, 2000.