Eilenberg lectures - Fall 2023

Some new geometric structures in the Langlands program

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CHAPTER 1

First lecture - Sept 12

1. The local Langlands correspondence

1.1. Notations. Fix a prime number p. We need the following datum

- E is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and uniformizer π
- We fix an algebraic closure $\overline{E}$ of E and let
$$\Gamma_E = \mathrm{Gal}(\overline{E}|E)$$
and
$$W_E \subset \Gamma_E$$
be the associated Weil group of elements of Γ_E acting as Frob_q^n for some $n \in \mathbb{Z} \subset \widehat{\mathbb{Z}}$ on the residue field.
- G is a reductive group over E
- We fix some $\ell \neq p$ and consider $\overline{\mathbb{Q}}_\ell$ an algebraic closure of $\mathbb{Q}_\ell$

We let
$$^L G = \widehat{G} \rtimes \Gamma_E$$
be *the associated L-group* over $\mathbb{Z}$ (seen as a pro-algebraic group). Here $\widehat{G}$ is a split reductive group over $\mathbb{Z}$ equipped with an action of Γ_E factorizing through an open subgroup of Γ_E.

Example 1.1. *(1) If $G = T$ is a torus then $\widehat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{G}_m$ with the Γ_E action deduced from the one on $X^*(T)$*
(2) If $G = \mathrm{GL}_{n/E}$ then $\widehat{G} = \mathrm{GL}_n$ with trivial Γ_E action
(3) If $G = \mathrm{SL}_{n/E}$ then $\widehat{G} = \mathrm{PGL}_n$ with trivial Γ_E action
(4) If $K|E$ is a quadratic extension with Galois group $\{\mathrm{Id}, *\}$, $A \in M_n(K)$ is hermitian non-degenerate, i.e. satisfies ${}^t A^* = A$ and $\det(A) \neq 0$, the associated unitary group G such that $G(E) = \{B \in \mathrm{GL}_n(K) \mid BA\, {}^t B^* = A\}$ satisfies $\widehat{G} = \mathrm{GL}_n$ with the action of Γ_E factorizing through Gal$(K|E)$ and where the non-trivial element of the Galois group acts as $g \mapsto w\, {}^t g^{-1} w$ where $w = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}$

1.2. The local Langlands correspondence.

1.2.1. *Smooth representations.* Let Λ be a $\mathbb{Z}[\frac{1}{p}]$-algebra. Recall the following definition.

Definition 1.2. *A smooth representation of $G(E)$ with coefficients in Λ is a Λ-module M equipped with a linear action of $G(E)$ such that the stabilizer of any vector is open in $G(E)$. We note*

$$\mathrm{Rep}_\Lambda(G(E))$$

for the category of smooth representations with coefficients in Λ.

Let

$$\mathscr{C}(G(E), \Lambda)$$

be the Λ-module of locally constant with compact support functions on $G(E)$ with coefficients in Λ. Let

$$\mathscr{H}_\Lambda(G(E)) = \mathrm{Hom}_\Lambda(\mathscr{C}(G(E), \Lambda), \Lambda)$$

be *the Hecke convolution algebra* of distributions on $G(E)$ with coefficients in Λ that are smooth with compact support. The choice of a Haar measure μ on $G(E)$ with values in $\mathbb{Z}[\frac{1}{p}]$ defines an isomorphism

$$\begin{array}{ccc} \mathscr{C}(G(E), \Lambda) & \xrightarrow{\sim} & \mathscr{H}_\Lambda(G(E)) \\ f & \longmapsto & f\mu \end{array}$$

where the ring structure on $\mathscr{C}(G(E))$ is now given by $(f * g)(x) = \int_{G(E)} f(xy^{-1})g(y)d\mu(y)$. For each $K \subset G(E)$ an open pro-p subgroup there is associated an *idempotent*

$$e_K \in \mathscr{H}_\Lambda(G(E))$$

given by $\langle e_K, \varphi\rangle = \int_K \varphi$ where, in this formula, the integration on K is with respect to the Haar measure with volume 1. In other words, $e_K = \frac{1}{\mu(K)}\mathbf{1}_K \in \mathscr{C}(G(E))$ via the preceding identification. Then, one has $e_K * e_{K'} = e_K$ if $K \subset K'$ and

$$\mathscr{H}_\Lambda(G(E)) = \bigcup_K \underbrace{e_K * \mathscr{H}(G(E), \Lambda) * e_K}_{\mathscr{H}_\Lambda(K\backslash G(E)/K)}$$

where $\mathscr{H}(K\backslash G(E)/K)$ is the Hecke algebra of K-bi-invariant distributions on $G(E)$ with compact support.

To any $\pi \in \mathrm{Rep}_\Lambda(G(E))$ with associated Λ-module M_π, one can associate a module over $\mathscr{H}(G(E), \Lambda)$ by setting for $m \in M_\pi$ and $T \in \mathscr{H}(G(E), \Lambda)$,

$$T.m = \int_{G(E)} \pi(g).m \; dT(g).$$

One then has

$$e_K.M_\pi = M_\pi^K$$

as an $\mathscr{H}(K\backslash G(E)/K, \Lambda)$-module. This induces an equivalence

$$\{\text{smooth rep. of } G(E) \text{ wt. coeff. in } \Lambda\} \xrightarrow{\sim} \{\mathscr{H}_\Lambda(G(E))\text{-modules } M \text{ s.t. } M = \cup_K e_K.M\}.$$

One verifies that if Λ is a field and K is compact open with order invertible in Λ this induces an equivalence

$$\{\pi \in \mathrm{Rep}_\Lambda(G(E)) \text{ irreducible s.t. } \pi^K \neq 0\} \xrightarrow{\sim} \{\text{irreducible } \mathscr{H}_\Lambda(K\backslash G(E)/K)\text{-modules }\}.$$

1.2.2. *Langlands parameters.* The local Langlands correspondence seeks to attach to any *irreducible* $\pi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ a *Langlands parameter*

$$\varphi_\pi : W_E \longrightarrow {}^LG(\overline{\mathbb{Q}}_\ell).$$

Here the terminology "Langlands parameter" means

- that the composite of φ_π with the projection to Γ_E is the canonical inclusion $W_E \subset \Gamma_E$ i.e. φ_π is given by a 1-cocycle $W_E \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$,
- that moreover this cocycle takes values in $\widehat{G}(L)$ where L is a finite degree extension of $\mathbb{Q}_\ell$,
- that this cocycle with values in $\widehat{G}(L)$ is continuous.

Remark 1.3. *There's a way to make this notion of Langlands parameter independent of the choice of the ℓ-adic topology. In fact, Grothendieck's ℓ-adic monodromy theorem ("any ℓ-adic representation is potentially semi-stable") applies in this context and a Langlands parameter $\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ as before is in fact the same as a couple (ρ, N) where*

- *$\rho : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is a Langlands parameter that is trivial on an open sub-group of W_E,*
- *$N \in \mathfrak{g}_{\overline{\mathbb{Q}}_\ell}(-1)$ is nilpotent and satisfies: $\forall \tau \in W_E,\ \mathrm{Ad}\,\rho(\tau).N = q^{v(\tau)}N$ where τ acts as $\mathrm{Frob}_q^{v(\tau)}$ on the residue field.*

The couples

$$(\rho, N)$$

are the so-called **Weil-Deligne parameters**. *There is a 1-cocycle $t_\ell : W_E \to \mathbb{Z}_\ell(1)$ sending τ to $\left(\tau(\pi^{1/\ell^n})/\pi^{1/\ell^n}\right)_{n\geq 1}$. The correspondence sends (ρ, N) to the parameter φ such that for $\tau \in W_E$,*

$$\varphi(\tau) = \rho(\tau)\exp(t_\ell(\tau)N) \rtimes \tau.$$

Nevertheless, since we fix a prime number ℓ in our work with Scholze we prefer to give a formulation using the ℓ-adic topology. This is justified by the fact that we construct such parameters over $\overline{\mathbb{F}}_\ell$ too and *our correspondence is compatible with mod ℓ reduction.*

One last remark: φ_π is only defined up to $\widehat{G}(\mathbb{Q}_\ell)$-conjugation i.e. we see it as an element of $H^1(W_E, \widehat{G}(\overline{\mathbb{Q}}_\ell))$. Up to now the local Langlands correspondence is a map

$$\mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(G(E))/\sim\ \longrightarrow\ \{\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)$$

i.e. a map between isomorphism classes of object. We will later see *this correspondence has some categorical flavors* (and this is quite important since at the end we formulate a real categorical local Langlands correspondence with Scholze) but up to now we deal with objects up to isomorphisms

1.2.3. *What to expect from the local Langlands correspondence.* Here is what we expect from the local Langlands correspondence.

(1) **Frobenius semi-simplicity** First, there is one condition on φ_π: this has to be *Frobenius semi-simple* in the sense that the associated couple (ρ, N) has to be such that for all τ, $\rho(\tau)$ is semi-simple (i.e. $\rho(\tau)$ is semi-simple for a τ satisfying $v(\tau) = 1$).

(2) **Finiteness of the L-packets** The fibers of $\{\pi\} \mapsto \{\varphi_\pi\}$ are finite: those are the so-called L-packets

(3) **Description of the image** When G is quasi-split the correspondence $\{\pi\} \mapsto \{\varphi_\pi\}$ should be surjective. For other G, there is so-called relevance condition so that a parameter $\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)$ is isomorphic to some φ_π if and only if as soon as φ factorizes (up to $\widehat{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy) through some parabolic subgroup ${}^L P(\overline{\mathbb{Q}}_\ell)$ where P is a parabolic subgroup of G^* then P transfers to G.

   For example: if $G = D^\times$ where D is a central division algebra over E with $[D : E] = n^2$ then a Langlands parameter $\varphi : W_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) = \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is relevant if and only if φ, as a linear representation of W_E, is indecomposable.

(4) **Compatibility with local class field theory** If $G = T$ is a torus class field theory gives an isomorphism of groups

$$\mathrm{Hom}(T(E), \overline{\mathbb{Q}}_\ell^\times) \xrightarrow{\sim} H^1(W_E, {}^L T(\overline{\mathbb{Q}}_\ell))$$

   this has to be the cloal Langlands correspondence for tori. Typically, when T is a spli torus, there is an Artin reciprocity isomorphism

$$T(E) \xrightarrow{\sim} W_E^{ab} \otimes_{\mathbb{Z}} X_*(T)$$

   deduced from

$$\mathrm{Art}_E : E^\times \xrightarrow{\sim} W_E^{ab},$$

   and this isomorphism induces the local Langlands correspondence for T.

(5) **Compatibility with the unramified local Langlands correspondence (Satake isomorphism)** If G is unramified, K is hyperspecial, after the choice of a square root of q in $\overline{\mathbb{Q}}_\ell$, there is a *Satake isomorphism* given by a *constant term map*

$$\mathscr{H}(K\backslash G(E)/K) \xrightarrow{\sim} \mathscr{H}(T(\mathcal{O}_E)\backslash T(E)/T(\mathcal{O}_E))^W$$

   where T is an unramified torus coming from an integral model associated to the choice of K. If $A \subset T$ is the maximal split torus inside T then

$$\mathscr{H}(T(\mathcal{O}_E)\backslash T(E)/T(\mathcal{O}_E))^W = \mathscr{H}(A(\mathcal{O}_E)\backslash A(E)/A(\mathcal{O}_E))^W$$

   that is identified with

$$\overline{\mathbb{Q}}_\ell[X_*(A)]^W = \overline{\mathbb{Q}}_\ell\big[X^*(\widehat{A})\big]^W.$$

   If π is such that $\pi^K \neq 0$ then the irreducible module π^K over the spherical Hecke algebra thus defines a character

$$\overline{\mathbb{Q}}_\ell\big[X^*(\widehat{A})\big]^W \longrightarrow \overline{\mathbb{Q}}_\ell$$

that is to say an element of $\widehat{A}(\overline{\mathbb{Q}}_\ell)/W$. One can prove that this is the same as an element of

$$\{\text{unramified (semi-simple) } \varphi : W_E/I_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\} \,/\, \widehat{G}(\overline{\mathbb{Q}}_\ell)$$

(6) **Compatibility with Kazhdan-Lusztig depth 0 local Langlands**

If G is split and I is an Iwahori subgroup of $G(E)$ then the category

$$\mathrm{Rep}^I_{\overline{\mathbb{Q}}_\ell}(G(E))$$

of $\pi \in \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ generated by π^I form a block in $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ in the sense that there is an indecomposable idempotent e in the Bernstein center of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E))$ such that

$$e.\,\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(G(E)) = \mathrm{Rep}^I_{\overline{\mathbb{Q}}_\ell}(G(E)).$$

This is the so-called central block. This category is then identified with the category of modules over the Iwahori-Hecke algebra

$$\mathscr{H}(I\backslash G(E)/I).$$

The identification of this Iwahori-Hecke algebra with the equivariant K-theory of the Steinberg variety has allowed Kazhdan and Lusztig to give a parametrization of irreducible $\mathscr{H}(I\backslash G(E)/I)$-modules as couples (s, N) where $s \in \widehat{G}(\overline{\mathbb{Q}}_\ell)$ is semi-simple and $N \in \widehat{\mathfrak{g}}_{\overline{\mathbb{Q}}_\ell}$ is nilpotent and satisfies $\mathrm{Ad}(s).N = qN$. We ask that this is the local Langlands correspondence in this case.

(7) **Compatibility up to semi-simplification with parabolic induction**

We say a parameter φ is semi-simple if the associated Weil-Deligne Langlands parameter (ρ, N) is such that $N = 0$. Equivalently, $\varphi_{|I_E}$ is trivial on an open subgroup. For a parameter φ we can define φ^{ss} its semi-simplification. Then, if P is a parabolic subgroup with Levi subgroup M we ask the following: for π an irreducible smooth representation of $M(E)$, if π' is an irreducible subquotient of the finite length representation

$$\mathrm{Ind}_{P(E)}^{G(E)} \pi$$

(normalized parabolic induction), then

$$\varphi^{ss}_{\pi'}$$

is the composite of φ^{ss}_π with the inclusion ${}^L M(\overline{\mathbb{Q}}_\ell) \hookrightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$.

Let us remark that, of course, this is false without the semi-simplification since the Steinberg representation of $\mathrm{GL}_n(E)$ and the trivial one do not have the same Langlands parameters.

(8) **Categorical flavor: description of supercuspdial L-packets**

We are now introducing some categorical flavor inside the Langlands parameters: we are not looking at the set quotient

$$\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)$$

but the quotient as a groupoid

$$\big[\,\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)\,\big],$$

and thus

$$\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell) = \pi_0\big[\,\{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell)\}/\widehat{G}(\overline{\mathbb{Q}}_\ell)\,\big].$$

Suppose G is quasi-split (we will see later, following the work of Vogan, Kottwitz and Kaletha what to do in the non-quasi-split case). For a parameter φ we define

$$S_\varphi = \{g \in \widehat{G}(\overline{\mathbb{Q}}_\ell) \mid g\varphi g^{-1} = \varphi\}.$$

This is the automorphism group of φ in the preceding groupoid. There is always an inclusion

$$Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E} \subset S_\varphi.$$

We say that φ is cuspidal if it is semi-simple and $S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}$ is finite. We say a packet is supercuspidal if all of its elements are supercuspidal. Then

$$\{\text{supercuspidal L-packets}\} \xrightarrow{\sim} \{\varphi : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell) \text{ cuspidal }\}/\widehat{G}(\overline{\mathbb{Q}}_\ell).$$

Moreover, the choice of a Whittaker datum defines a bijection for φ a cuspidal parameter

$$\mathrm{Irr}(\underbrace{S_\varphi/Z(\widehat{G})(\overline{\mathbb{Q}}_\ell)^{\Gamma_E}}_{\text{finite group}}) \xrightarrow{\sim} \text{L-packet associated to } \varphi$$

where the trivial representation should correspond to the unique generic (with respect to the choice of the Whittaker datum) representation of the L-packet.

(9) **Local global compatibility**

Let K be a number field and Π be an algebraic automorphic representation of G where now G is a reductive group over K. Conjecturally, Π_f is defined over a number field as a smooth representation of $G(\mathbb{A}_f)$. Let us fix an embedding of this number field inside $\overline{\mathbb{Q}}_\ell$. Then one should be able to attach to Π an ℓ-adic Langlands parameter

$$\varphi_\Pi : \mathrm{Gal}(\overline{K}|K) \longrightarrow {}^L G(\overline{\mathbb{Q}}_\ell).$$

For a place v of K dividing $p \neq \ell$,

$$\varphi_{\Pi\,|\,W_{K_v}}$$

depends only on Π_v and is given up to conjugation by

$$\varphi_{\Pi_v}.$$

2. Background on the global Langlands correspondence and global Langlands parameters

Let G be a reductive group over a number field K. Let Π be an automorphic representation of G i.e. an irreducible sub-quotient of the space of automorphic forms on G. As an abstract representation

$$\Pi \simeq \bigotimes_v \Pi_v$$

where v goes through the places of K. If $v|\infty$, the local Langlands correspondence is known for Π_v is known and we can define

$$\varphi_{\Pi_v} : W_{K_v} \longrightarrow^L G_{\mathbb{C}}.$$

There is a natural morphism

$$\mathbb{C}^\times \longrightarrow W_{K_v}$$

that is an isomorphism if $K_v \simeq \mathbb{C}$ and fits into a non-split exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{K_v} \longrightarrow \mathrm{Gal}(\mathbb{C}|\mathbb{R}) \longrightarrow 1$$

if $K_v \simeq \mathbb{R}$.

Definition 2.1. *An automorphic representation* Π *of* G *is algebraic if for all* $v|\infty$, $\varphi_{\Pi_v|\mathbb{C}^\times} : \mathbb{C}^\times \longrightarrow \widehat{G}(\mathbb{C})$ *is algebraic i.e. is given by an algebraic morphism* $\mathbb{S}_\mathbb{C} \to \widehat{G}_\mathbb{C}$ *where* $\mathbb{S}$ *is Deligne's torus* $\mathrm{Res}_{\mathbb{C}|\mathbb{R}}\,\mathbb{G}_m$ *via the inclusion* $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$.

It is the same as to ask that for all $v|\infty$, Π_v has the same infinitesimal character as the one of an algebraic irreducible finite dimension representation of the algebraic group $G_{\overline{K_v}}$ with coefficients in $\mathbb{C}$.

Conjecturally there exists a global Langlands group

$$\mathscr{L}_K$$

that is a locally compact topological group sitting in an exact sequence

$$1 \longrightarrow \mathscr{L}_K^\circ \longrightarrow \mathscr{L}_K \longrightarrow \mathrm{Gal}(\overline{K}|K) \longrightarrow 1$$

and with an identification

$$\mathscr{L}_K/(\mathscr{L}_K^\circ)' = W_K$$

the global Weil group. Moreover, on expects the following.

Conjecture 2.2. *The following is expected:*

(a) *To each automorphic representation* Π *of* G *one can associate a Langlands parameter*

$$\varphi_\Pi : \mathscr{L}_K \longrightarrow {}^L G_{\mathbb{C}}$$

compatibly with the local Langlands correspondence at archimedean places and the unramified one at almost all finite places

(b) *If* Π *is algebraic then* Π_f *is defined over a number field inside* $\mathbb{C}$ *and to the choice of an embedding of such a number field inside* $\overline{\mathbb{Q}}_\ell$ *is associated an* ℓ*-adic Langlands parameter*

$$\varphi_{\Pi,\ell} : Gal(\overline{K}|K) \longrightarrow {}^L G_{\overline{\mathbb{Q}}_\ell}$$

(c) *The Tannakian category of continuous representations of* $\mathscr{L}_K$ *on finite dimensional* $\mathbb{C}$*-vector spaces that are algebraic is identified with the category of Grothendieck motives for numerical equivalence with* $\mathbb{C}$ *coefficients.*

This is known for tori when we consider the category of CM-motives for absolute Hodge cycles.

The construction of the ℓ-adic Langlands parameters is know for cohomological automorphic representations of GL_n. Other cases are known using the cohomology of Shimura varieties.

For example, if $f = \sum_{n\geq 1} a_n q^n$ is a normalized weight $k \geq 1$ holomorphic modular form for $\Gamma_0(N)$ that is new and an Hecke eigenvector of the Hecke operators $(T_p)_{p\nmid N}$ then one can associate (Shimura, Deligne, Deligne-Serre) a Galois representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$$

such that for $p \nmid N$, that characteristic polynomial of $\rho_f(\mathrm{Frob}_p)$ is $X^2 - a_p X + p^{k-1}$.

3. What we do with Scholze

We prove the following theorem.

Theorem 3.1 (F.-Scholze). *For* ℓ *a good prime with respect to* G *(any* ℓ *if* $G = \mathrm{GL}_n$*,* $\ell \neq 2$ *for classical groups) there exists a monoidal action of the category of perfect complexes*

$$\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})$$

on

$$D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$$

where $\mathrm{LocSys}_{\widehat{G}} \to Spec(\mathbb{Z}[\frac{1}{p}])$ *is the moduli space of Langlands parameter, an algebraic stack locally complete intersection of dimension* 0 *over* $Spec(\mathbb{Z}[\frac{1}{p}])$.

As a consequence we can construct the semi-simple local Langlands correspondence

$$\pi \mapsto \varphi_\pi^{ss}$$

for any reductive group over E, over $\overline{\mathbb{F}}_\ell$ and $\overline{\mathbb{Q}}_\ell$ (and compatibly with mod ℓ reduction).

As for now the statement of the local Langlands conjecture is the following.

Conjecture 3.2 (Categorical local Langlands). *Suppose G is quasi-split and fix a Whittaker datum (B, ψ). Suppose ℓ is a good prime. There exists an equivalence of stable ∞-categories*

$$\mathcal{D}^b_{coh}(\mathrm{LocSys}_{\widehat{G}/\overline{\mathbb{Z}}_\ell})_{nilp.ss.supp} \xrightarrow{\sim} \mathcal{D}_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega$$

compatible with the preceding spectral action and sending the structural sheaf $\mathcal{O}$ to the Whittaker sheaf.

The goal of those lectures is to explain how after 20 years of work, starting from the classical local Langlands correspondence in terms of parameters of smooth irreducible representations as in the work of Harris-Taylor, we arrived at such a statement and what are those geometric objects showing up in the preceding statement, starting with the so-called Lubin-Tate spaces continuing with Rapoport-Zink spaces, Hodge-Tate periods, the curve and so on.