LECTURE NOTES AND EXERCISES FOR THE 2023-2024 COURSE "THE COUNT OF INSTANTONS" AT COLUMBIA UNIVERSITY DEPARTMENT OF MATHEMATICS

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Abstract. The notation □`text`■ denotes the exercises.
The notation ♠`text`♢ denotes the definitions.

1. Review of classical mechanics

The arena of most (but not all) of *Classical mechanics* is the world of
♠ *symplectic manifolds* $(\mathcal{P}, \omega)$, where $\omega \in Z^2(\mathcal{P})$ is a closed, non-degenerate: $\omega^{\wedge m} \neq 0$, two-form on a smooth manifold $\mathcal{P}$ of dimension

$$\dim \mathcal{P} = 2m \tag{1.1}$$

supplemented by a choice of the *Hamiltonian* $H \in C^\infty(\mathcal{P})$ ♢.

$$(\mathcal{P}, \omega, H) \text{ , with } , \tag{1.2}$$

The symplectic form makes the ring $A = C^\infty(\mathcal{P})$ of smooth functions a Lie algebra, with the ♠ *Poisson bracket* given by

$$\{f, g\} = \iota_\pi (df \wedge dg) \ , \tag{1.3}$$

with the Poisson bi-vector $\pi = \omega^{-1}$. ♢
□ The closedness $d\omega = 0$ implies the Jacobi identity for $\{\cdot, \cdot\}$:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \qquad ■ \tag{1.4}$$

Examples:

(1) $\mathcal{P} = T^*X$ where X is any manifold,

$$\omega = d\theta \tag{1.5}$$

with $\theta \in \Omega^1(T^*X)$-canonical 1-form

$$\theta_{(p,q)}(\xi) = p(\pi_* \xi) \tag{1.6}$$

where $q \in X$, $p \in T^*_q X$, $\xi \in T_{(p,q)} T^*X$, $\pi : T^*X \to X$ is the projection, and $\pi_* \xi \in T_q X$ is the projection of the vector tangent to T^*X to the base tangent vector.

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(2) Let (M, ω_M) be a symplectic manifold. Then $(\mathcal{P} = T^*M, \omega_{\mathcal{P}} = d\theta + k\pi^*\omega_M)$ defines a family of symplectic manifolds. The corresponding evolution is sometimes called a *motion in magnetic field.*

(3) Let G be a simple Lie group, $\mathfrak{g} = LieG$ its Lie algebra, and $\mathfrak{g}^*$ the dual space. Let $\xi \in \mathfrak{g}$. Define $\mathcal{P} = \mathcal{O}_\xi := \{Ad^*_g\xi \,|\, g \in G\}$ be the coadjoint orbit (of ξ). It carries the canonical Kirillov-Kostant symplectic form. Let us define it through the Poisson brackets of functions on $\mathcal{P}$:

$$\{f_1, f_2\}(x) = x\left([df_1, df_2]\right) \tag{1.7}$$

Here the functions $f_{1,2} : \mathfrak{g}^* \to \mathbb{R}$ have differentials $df_{1,2}$ which, at the point $x \in \mathcal{P} \subset \mathfrak{g}^*$ are the linear functions on $T_x\mathfrak{g}^* \approx \mathfrak{g}^*$, i.e. (for finite dimensional vector spaces $V \approx V^{**}$) elements $v_{1,2} \in \mathfrak{g}$. We then evaluate x, as a linear function on $\mathfrak{g}$, on the commutator $[v_1, v_2]$. □ Show (1.7) is invertible i.e. corresponds to a symplectic form ■.

(4) Specifically, let $G = SU(N)$. We can view G as a subgroup of the group $U(N)$ of automorphisms of the N-dimensional vector space $\mathbf{N} \approx \mathbb{C}^N$ endowed with a hermitian form, i.e. sesquilinear non-degenerate pairing $\langle \cdot, \cdot \rangle : \mathbf{N} \times \mathbf{N} \to \mathbb{C}$, obeying

$$\langle xv_1, yv_2 \rangle = \bar{x}y\langle v_1, v_2 \rangle \,, \ \langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \tag{1.8}$$

for any $x, y \in \mathbb{C}$, $v_1, v_2 \in V$. So, $g \in GL(\mathbf{N})$ belongs to $U(N)$ if for any $v_1, v_2 \in V$

$$\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle \tag{1.9}$$

The subgroup G is singled out by the condition $det(g) = 1$. In other words, special unitary transformations preserve a volume form $\Omega \in \Lambda^N\mathbf{N}^*$, in addition to the hermitian form. Now recall that the operator $A \in End(V)$ is called hermitian, if for any $v_1, v_2 \in V$

$$\langle v_1, Av_2 \rangle = \langle Av_1, v_2 \rangle \tag{1.10}$$

Let us choose a basis $\mathbf{e}_1, \ldots, \mathbf{e}_N$, orthonormal with respect to the hermitian form:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{i,j} \tag{1.11}$$

In this basis the operators g, A have the associated matrices $\|g_{i\bar{j}}\|, \|A_{i\bar{j}}\|$,

$$g_{i\bar{j}} = \langle \mathbf{e}_i, g\mathbf{e}_j \rangle \,, \qquad A_{i\bar{j}} = \langle \mathbf{e}_i, A\mathbf{e}_j \rangle \tag{1.12}$$

(the bar on $\bar{j}$ signifies the different role $\mathbf{e}_i$ and $\mathbf{e}_j$ play in the right hand side of the equation).

□ Show the unitarity of g and hermiticity of A is equivalent to the set of equations

$$
\begin{aligned}
& gg^{\dagger} = \mathbf{1_N} \;\Leftrightarrow\; \sum_{j=1}^{N} g_{i\bar{j}} g_{j\bar{k}} = \delta_{i\bar{k}} \,, \qquad i, \bar{k} = 1, \ldots, N \\
& A = A^{\dagger} \;\Leftrightarrow\; A_{i\bar{j}} = \overline{A_{j\bar{i}}} \qquad \blacksquare
\end{aligned} \tag{1.13}
$$

The Lie algebra $\mathfrak{g} = LieU(N)$ is the vector space of all anti-hermitian operators in $\mathbf{N}$:

$$
B \in \mathfrak{g} \Leftrightarrow \langle v_1, Bv_2 \rangle + \langle Bv_1, v_2 \rangle = 0 \tag{1.14}
$$

Of course, if B is antihermitian, then $A = \mathrm{i}B$ is hermitian and vice versa. The Lie algebra of $SU(N)$ is a subspace of all traceless antihermitian matrices. Consider the set of all hermitian operators with fixed eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$:

$$
\mathcal{O}_{\lambda_1, \ldots, \lambda_N} = \left\{ A \,|\, \mathrm{Det}(\lambda - A) = \prod_{i=1}^{N} (\lambda - \lambda_i) \right\} \tag{1.15}
$$

Using the pairing

$$
\langle A, B \rangle := \mathrm{i}\mathrm{tr}_{\mathbb{N}} AB \tag{1.16}
$$

we identify $\mathfrak{g}^*$ with the space of Hermitian operators in $\mathbf{N}$. Thus, $\mathcal{O}_{\lambda_1, \ldots, \lambda_N} \subset \mathfrak{g}^*$ is a coadjoint orbit of $U(N)$. To make it into the coadjoint orbit of $SU(N)$ we need to descend to the quotient of the space of all Hermitian matrices by the action of $\mathbb{R}$ of shifts by a scalar operator:

$$
B \sim B + b \cdot \mathbf{1_N} \,, \qquad b \in \mathbb{R} \tag{1.17}
$$

We can fix the representative by demanding that the B operators are also tracefree, $\mathrm{tr}B = 0$. Thus, let

$$
\lambda_1 + \ldots + \lambda_N = 0 \tag{1.18}
$$

Then, $\mathcal{O}_{\lambda_1, \ldots, \lambda_N} \subset \mathfrak{su}(N)^*$ is a coadjoint orbit of $SU(N)$. *Flag varieties, Grassmanians, projective spaces.*

(5) As is customary in theoretical physics we shall take the above definitions and try our best in extending them to the infinite-dimensional settings. The loop space $LX = \mathrm{Maps}\left(S^1, X\right)$ of smooth maps of a circle to a Riemannian manifold (X, g_X) carries a closed two-form Ω_{LX}. At some loop $\gamma \in LX$ its value on a pair of vectors $\xi_1, \xi_2 \in \Gamma(S^1, \gamma^* TX)$ is given by:

$$
\Omega_{LX}(\xi_1, \xi_2) = \int_{S^1} \gamma^* g_X(\xi_1, \nabla \xi_2) \tag{1.19}
$$

where ∇ is the pull-back by γ of the Levi-Civita connection on TX defined by the metric g_X. □ Is Ω_{LX} a symplectic form? ■.

(6) This example can be generalized to the case of $\mathcal{P} = Maps(M, X)$, where M is a compact manifold of dimension n, endowed with a closed $n-1$-form ν_M. We define

$$\Omega_{X^M}(\xi_1, \xi_2) = \int_M \nu_M \wedge \gamma^* g_X(\xi_1, \nabla \xi_2) \tag{1.20}$$

(7) Let (M, μ_M) be a compact manifold endowed with a volume form $\mu_M \in \Omega^{\dim(M)}(M)$, $\mu_M \neq 0$, and let (X, ω_X) be a symplectic manifold. Define $\mathcal{P} = Maps(M, X)$, and endow it with the symplectic form, s.t. at $\gamma : M \to X$ and $\xi_1, \xi_2 \in \Gamma(M, \gamma^* TX)$

$$\Omega_{X^M}(\xi_1, \xi_2) = \int_M \mu_M \, \gamma^* \omega_X(\xi_1, \xi_2) \tag{1.21}$$

1.1. **Hamilton equations.** Now let us put the function $H \in A$ to a good use. The differential dH is a 1-form on $\mathcal{P}$. Define the Hamiltonian vector field V_H by:

$$\iota_{V_H} \omega = dH \Leftrightarrow V_H = \iota_{\omega^{-1}} dH \tag{1.22}$$

Examples:

(1) Let X be any manifold and $v \in Vect(X)$ a vector field. Let $\mathcal{P} = T^*X$ with $\omega = d\theta$, and $H(p, q) = p(v(q))$, $q \in X$, $p \in T^*_q X$. The corresponding vector field V_H covers the vector field v on X. □ Compute V_H. ■

(2) Let (X, g) be a Riemannian manifold. We view the metric g as a smooth map $TX \to T^*X$. Then $(\mathcal{P} = T^*X, \omega = d\theta, H)$ with

$$H = \frac{1}{2}(p, g^{-1} p) \tag{1.23}$$

defines the Hamiltonian system, covering the *geodesic flow* on X [1]

(3) Again, let (X, g) be a Riemannian manifold and $U \in C^\infty(X)$. Then

$$H(p, q) = \frac{1}{2}(p, g(q)^{-1} p) + U(q) \tag{1.26}$$

describes a particle on X moving in the field of the potential U.

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[1]♠ A geodesic curve on a Riemannian manifold (X, g) is the extremum of the functional

$$L = \int_U \sqrt{g(\dot{\ell}, \dot{\ell})} \, dt \tag{1.24}$$

on the space $Maps(U, X)/Diff_+(U)$ of oriented curves in X. Here $U \subset \mathbb{R}_t$ is a connected domain, and $\ell : U \to X$ a smooth map. One can impose Dirichlet boundary conditions: $\ell(\partial U) = \{x_{\mathbf{s}}, x_{\mathbf{t}}\}$ with two points $x_{\mathbf{s}}, x_{\mathbf{t}} \in X$. Analogously, a minimal surface in X is the extremum of the functional

$$A = \int_U \sqrt{g(\dot{\ell}, \dot{\ell}) g(\ell', \ell') - g(\dot{\ell}, \ell')^2} \, dt ds \tag{1.25}$$

on the space $Maps(U, X)/Diff_+(U)$ of oriented surfaces in X. Here $U \subset \mathbb{R}^2_{t,s}$ is a connected domain, and $\ell : U \to X$ a smooth map. One can impose Dirichlet boundary conditions: $\ell(\partial U) = \gamma$, $\gamma \subset X$, or Neumann boundary conditions: $\nabla_n \ell|_{\partial U} = 0$. ♢

(4) An important special case of the system above is ♠ the generalized harmonic oscillator (or a system of m oscillators): $X = T^*Q$, $Q = \mathbb{R}^m$, $\omega = \sum_{i=1}^m dp_i \wedge dq^i$,

$$H = \frac{1}{2}\sum_{i,j=1}^m g^{ij}p_ip_j + \sum_{i,j=1}^m K_i^j p_j q^i + \frac{1}{2}\sum_{i,j=1}^m U_{ij}q^iq^j \tag{1.27}$$

where we assume the non-degenerate positive definite metric g_{ij} on Q, whose inverse g^{ij} appears as the *kinetic term* in (1.27). We also assume the second metric U_{ij} on the same Q, defining the *potential term*. The cross-term (p, Kq) depends on a choice of linear operator $K : V \to V$. The operator K can be decomposed as a sum of g-symmetric and g-antisymmetric parts:

$$K = K^s + K^a\,,\ gK^s = (K^s)^t g\,,\ gK^a = -(K^a)^t g \tag{1.28}$$

where we viewed the metric g as the symmetric map $g : V \to V^*$, [2] $g^t = g$. The symmetric part K^s can be eliminated from H by a canonical transformation:

$$p \mapsto p - g(K^s q) \tag{1.29}$$

$$\sum d\left(g_{il}K^l_m q^m\right) \wedge dq^i = 0 \tag{1.30}$$

What is the meaning of K^a? Define $\Omega = gK^a$, $\Omega^t = -\Omega$. Then we can re-write the Hamiltonian as [3]:

$$H = \frac{1}{2}g^{ij}\left(p_i + \Omega_{ik}q^k\right)\left(p_j + \Omega_{jl}q^l\right) + \frac{1}{2}V_{ij}q^iq^j \tag{1.31}$$

where V is related to U in an obvious way. Finally, we can reinterpret (1.32) as the Hamiltonian evolution of a system of oscillators with standard kinetic and potential terms

$$\tilde{H} = \frac{1}{2}g^{ij}p_ip_j + \frac{1}{2}V_{ij}q^iq^j \tag{1.32}$$

but with a modified symplectic form:

$$\omega_{\rm mr} = dp_i \wedge dq^i + \frac{1}{2}\Omega_{ij}dq^idq^j \tag{1.33}$$

This modification is sometimes called *magnetic field-rotating frame.* □ Why? ■.

(5) Now let us assume X is endowed with a closed two-form F. Let $\mathcal{P} = T^*X$ with $\omega = d\theta + \pi^*F$. □ Compute V_H for H given by the Eq. (1.26). ■

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[2]♠ For a linear map $A : V_1 \to V_2$ we denote by A^t the canonical dual map $V_2^* \to V_1^*$, defined by $A^t\xi(v) = \xi(Av)$, for any $v \in V_1, \xi \in V_2^*$ ♢

[3]♠ Throughout the notes we adopt *Einstein's convention* except where it leads to confusion: $A^{i\cdots}_{\cdots}B^{\cdots}_{i\cdots} := \sum_i A^{i\cdots}_{\cdots}B^{\cdots}_{i\cdots}$ ♢

1.2. **Darboux coordinates, action-angle variables.** The local model of $\mathcal{P}$ is a domain in $T^*\mathbb{R}^m \approx \mathbb{R}^{2m}$, with $\omega = d\theta$, $\theta = \sum_{i=1}^m p_i dq^i$ with (q^i) coordinates on $\mathbb{R}^m$. We can change the coordinates by symplectic (canonical) diffeomorphisms. Let $g : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ preserves ω, then

$$g^*\theta - \theta = dS \tag{1.34}$$

for some function S. In coordinates:

$$\sum_{i=1}^{m} P_i dQ^i - p_i dq^i = dS \tag{1.35}$$

thus a function of $m + m$ variables *old and new q's* $S = S(Q, q)$ is all is needed to *generate* the symplectomorphism g (a beautifully non-invariant yet useful formalism):

$$P_i = \frac{\partial S}{\partial Q^i}\,, \qquad p_i = -\frac{\partial S}{\partial q^i} \tag{1.36}$$

Examples:

(1) Linear symplectomorpisms are generated by the quadratic functions

$$S = \sum_{i,j=1}^{m} \frac{1}{2} A_{ij} Q^i Q^j + B_{i|j} Q^i q^j + \frac{1}{2} C_{ij} q^i q^j \tag{1.37}$$

(2) In particular, time evolution of a system of harmonic oscillators is described by:

(3) Change of cartesian to polar coordinates on $\mathbb{R}^2$:

$$(p, q) \mapsto (A, \varphi) \;\; A = \frac{p^2 + q^2}{2}\,,\; q = \sqrt{2A}\cos(\varphi)\,,\; p = \sqrt{2A}\sin(\varphi) \tag{1.38}$$

$\square$ What is its generating function $S(\varphi, q)$? $\blacksquare$

1.3. **Symplectic quotients.** Suppose $(\mathcal{P}, \omega, H)$ is invariant under the action of a Lie group G. Moreover, we'll require the action of G to be Hamiltonian (this is automatic for simply-connected $\mathcal{P}$ $\square$ why? $\blacksquare$).

A diffeomorphism $g : \mathcal{P} \to \mathcal{P}$ is a symplectomorphism if it preserves ω:

$$g^*\omega = \omega \tag{1.39}$$

Infinitesimal symplectomorphism is a vector field $V \in Vect(\mathcal{P})$, such that

$$Lie_V \omega = 0 \leftrightarrow d(\iota_V \omega) = 0 \tag{1.40}$$

A Hamiltonian G-action on $\mathcal{P}$ associates to every $\xi \in \mathfrak{g}$, a Hamiltonian vector field V_ξ,

$$\iota_{V_\xi}\omega = dh_\xi \tag{1.41}$$

with some Hamiltonian function $h_\xi : \mathcal{P} \to \mathbb{R}$. Of course (1.41) defines h_ξ up to a constant. We can partly restrict the choice of a constant by requiring the map $\xi \mapsto h_\xi$ be linear. We have the homomorphism condition

$$[V_{\xi_1}, V_{\xi_2}] = V_{[\xi_1, \xi_2]} \tag{1.42}$$

for any $\xi_1, \xi_2 \in \mathfrak{g}$. We may want to require that

$$h_{[\xi_1,\xi_2]} = \{h_{\xi_1}, h_{\xi_2}\} := \omega(V_{\xi_1}, V_{\xi_2}) = \iota_{\omega^{-1}} dh_{\xi_1} \wedge dh_{\xi_2} \tag{1.43}$$

This is not always possible. For example, the abelian group $\mathbb{R}^2$ of translations acts on $\mathbb{R}^2$ preserving the constant volume 2-form $dp \wedge dx$. The Hamiltonians h_1, h_2 are equal to p, x, respectively. The corresponding vector fields commute, however $p, x = 1 \neq 0$.

Algebraically, the problem of finding the constants c_ξ adjusting h_ξ so as to obey (1.43) is the question of whether the Lie algebra $\mathfrak{g}$ has non-trivial second cohomology. Define

$$c(\xi_1, \xi_2) = \{h_{\xi_1}, h_{\xi_2}\} - h_{[\xi_1,\xi_2]} \tag{1.44}$$

As it stands $c(\xi_1, \xi_2)$ is a function on $\mathcal{P}$. However,

$$dc(\xi_1, \xi_2) = \iota_{[V_{\xi_1}, V_{\xi_2}]}\omega - \iota_{V_{[\xi_1,\xi_2]}}\omega = 0 \tag{1.45}$$

by Eq. (1.42). Thus, $c : \Lambda^2\mathfrak{g} \to \mathbb{R}$ defines a linear map. It obeys:

$$c([\xi_1, \xi_2], \xi_3) + c([\xi_2, \xi_3], \xi_1) + c([\xi_3, \xi_1], \xi_2) = \{h_{[\xi_1,\xi_2]}, h_{\xi_3}\} + cyclic = 0 \tag{1.46}$$

where we used the linearity of $\xi \mapsto h_\xi$ map and $\{h_{[\xi_1,\xi_2]}, \cdot\} = \{\{h_{\xi_1}, h_{\xi_2}\}, \cdot\}$.

1.3.1. *Cohomology of groups and algebras.* Let us pause to define an interesting cohomology theory. Let $\mathfrak{g}$ be a Lie algebra, and M a $\mathfrak{g}$-module, i.e. a vector space with the homomorphism $\rho : \mathfrak{g} \to End(M)$. For $\xi \in \mathfrak{g}$, $m \in M$ we denote $\rho(\xi)m$ simply by $\xi \cdot m$. Define $C^i(\mathfrak{g}, M)$ to be the space of all skew-symmetric polylinear functions $c_{(i)} : \Lambda^i\mathfrak{g} \longrightarrow M$. Define the differential $\delta : C^i(\mathfrak{g}, M) \to C^{i+1}(\mathfrak{g}, M)$ by

(1.47)

$$\delta c_{(i)} \left(\xi_1 \wedge \ldots \wedge \xi_{i+1}\right) = \sum_{j=1}^{i+1} (-1)^{j-1} \xi_j \cdot c_{(i)} \left(\xi_1 \wedge \ldots \wedge \widehat{\xi_j} \wedge \ldots \wedge \xi_{i+1}\right) + \sum_{1 \le a < b \le i+1} (-1)^{a+b} c_{(i)} \left([\xi_a, \xi_b] \wedge \xi_1 \wedge \ldots \widehat{\xi_a} \ldots \widehat{\xi_b} \ldots \wedge \xi_{i+1}\right)$$

$\square$ Check, that $\blacksquare$ it is a differential , $\delta^2 = 0$.

One defines $\spadesuit$ the cohomology groups

$$H^i(\mathfrak{g}, M) = \left(\mathrm{ker}\delta \cap C^i\right) / \left(\mathrm{im}\delta \cap C^i\right) \ \diamondsuit. \tag{1.48}$$

Let us now take, as example, $M = \mathbb{R}$ with a trivial action of $\mathfrak{g}$. The $H^2(\mathfrak{g}, \mathbb{R})$ group is the space of all skew-symmetric bilinear forms $c(\xi \wedge \eta)$ on $\mathfrak{g}$ which are closed under the differential

$$\delta c(\xi_1 \wedge \xi_2 \wedge \xi_3) = c([\xi_1, \xi_2] \wedge \xi_3) + c([\xi_2, \xi_3] \wedge \xi_1) + c([\xi_3, \xi_1] \wedge \xi_2) \tag{1.49}$$

modulo δ-exact forms $c \sim c + \delta b$, where $\delta b(\xi \wedge \eta) = b([\xi, \eta])$. For simple Lie algebras H^2 vanishes. For abelian Lie algebra $\mathfrak{a}$ the space of $\mathbb{R}$-valued

i-cocycles is the space of all skew-symmetric i-linear functions on $\mathfrak{a}$, since the commutators vanish implying the vanishing δ. In other words,

$$H^i(\mathfrak{a},\mathbb{R}) = \Lambda^i \mathfrak{a}^* \tag{1.50}$$

Any 2-cocycle $c \in Z^2(\mathfrak{g},\mathbb{R})$ defines a *central extension of* $\mathfrak{g}$, i.e. a new Lie algebra $\widetilde{\mathfrak{g}}$, which is equal to $\mathfrak{g} \oplus \mathbb{R}$ as a vector space, with the commutator defined by:

$$[(\xi_1, c_1), (\xi_2, c_2)] = ([\xi_1, \xi_2], c(\xi_1 \wedge \xi_2)) \tag{1.51}$$

However, not all such extensions are *non-trivial.* Indeed, $\xi \mapsto (\xi, b(\xi))$ embeds $\mathfrak{g}$ into $\widetilde{\mathfrak{g}}$ for linear function $b \in \mathfrak{g}^*$. Thus, only cocycles modulo coboundaries produce non-trivial, *new*, Lie algebras.

The most basic such extension is, in fact, an avatar of quantization. Given a symplectic vector space (V,ω) we define ♠ the Heisenberg algebra H_V to be the central extension of V viewed as abelian Lie algebra, corresponding to ω viewed as the Lie algebra-cohomology class $\omega \in H^2(V,\mathbb{R})$ ♢.

1.3.2. *Moment map.* Given a symplectic manifold $(\mathcal{P},\omega)$ and a Lie group G acting on $\mathcal{P}$ by Hamiltonian vector fields, define

$$\mathcal{P}^{\rm red} = \mathcal{P}//G = \mu^{-1}(\zeta)/G \tag{1.52}$$

where, assuming the vanishing of the obstruction class in $H^2(\mathfrak{g},\mathbb{R})$, the *moment map*

$$\mu : M \to \mathfrak{g}^* \tag{1.53}$$

is defined via:

$$\spadesuit\ \langle \mu(x), \xi\rangle \ = \ h_{\xi}(x)\, , \ \xi \in \mathfrak{g}\, , \ x \in M \ \diamondsuit \tag{1.54}$$

As discussed above, the possibility of adding a constant to h_ξ translates to a variety of possibilities for $\zeta \in (\mathfrak{g}^*)^G$in (1.52). For simple Lie group G only $\zeta = 0$ is possible, while for G with center there are many options for the level of the moment map. For example, take $M = \mathbb{R}^{2m}$, $\omega = \sum_{i=1}^m dp_i \wedge dx^i$, $G = U(1)$ acting via:

$$e^{\rm it} : (\mathbf{p}, \mathbf{x}) \mapsto (\mathbf{p}cos(t) - \mathbf{x}sin(t)\, , \ \mathbf{x}cos(t) + \mathbf{p}sin(t)) \tag{1.55}$$

The moment map

$$\mu = \frac{1}{2}\sum_{i=1}^{m} \left(p_i^2 + (x^i)^2\right) \tag{1.56}$$

The reduced phase space is empty for $\zeta < 0$, a point for $\zeta = 0$, and the remarkably important compact symplectic manifold of dimension $2(m-1)$ for $\zeta > 0$, the complex projective space $\mathbb{CP}^{m-1}$. As a bonus, it carries a complex structure, and so it can be described in complex terms only:

$$\mathbb{CP}^{m-1} = \left\{ (z_1, \ldots, z_m) \,|\, \mathbf{z} \neq 0\, , \ \mathbf{z} \sim u\mathbf{z}\, , \ u \in \mathbb{C}^{\times} \right\} = \left(\mathbb{C}^m \backslash 0\right)/\mathbb{C}^{\times} \tag{1.57}$$

In other words, we encountered the relation $M//G = M^s/G_{\mathbb{C}}$ which will be discussed in greater generality later.

2. Review of classical electromagnetism

Maxwell theory, or classical electromagnetism, is the infinite-dimensional version of classical mechanics, where the phase space $\mathcal{P}$ is the *field space*. We shall discuss it in somewhat artificial setting of compact spaces, which is easier to handle mathematically, and has applications to topology.

Let M^d be a compact d-dimensional manifold, endowed with Riemannian metric g. The metric defines the Hodge star operator $\star$ mapping p-forms to $d-p$-forms via the point-wise relation

$$\omega \star \omega = \mathrm{vol}_g \sum_{i_1<i_2<\ldots<i_p} \omega(e_{i_1}, \ldots, e_{i_p})\omega(f^{i_1}, \ldots, f^{i_p}) \tag{2.1}$$

where e_i, $i = 1, \ldots, m$ is any basis in T_xM, and f^i the associated orthogonal basis, such that

$$g(e_i, f^j) = \delta_i^j \tag{2.2}$$

for all $i, j = 1, \ldots, m$. The Eq. (2.1) endows Ω^p with the metric

$$(\omega, \eta) = \int_M \omega \wedge \star\eta = (\eta, \omega) \tag{2.3}$$

We shall glide over the finer details such as L^2 completions, Sobolev embeddings etc. which are needed to be able to operate with

$$d^* = \star d \star\,, \tag{2.4}$$

the operator $\Omega^i(M) \to \Omega^{i-1}(M)$, conjugate to d in the sense of the Hodge metric (2.3) on the space of differential forms.

2.1. $\mathbb{R}$-gauge p-form theory.

Our first approximation is to take

$$\mathcal{P} = T^*\mathcal{A}_{\mathbb{R}} // \mathcal{G}_{\mathbb{R}} \tag{2.5}$$

where $\mathcal{A}_{\mathbb{R}} = \Omega^p(M)$, $\mathcal{G}_{\mathbb{R}} = (\Omega^{p-1}(M)/Z^{p-1}(M))$ is the vector space of $p-1$-forms considered modulo closed ones, acting on $A_{\mathbb{R}}$ by

$$A \mapsto A + d\xi \tag{2.6}$$

for $A \in \mathcal{A}_{\mathbb{R}}$, $\xi \in \mathcal{G}_{\mathbb{R}}$. The symplectic form on $T^*\mathcal{A}_{\mathbb{R}}$ is traditionally written as

$$\Omega_{T^*\mathcal{A}_{\mathbb{R}}} = \int_M \delta E \wedge \delta A \tag{2.7}$$

where δ denotes the de Rham differential *in the space of fields* while d is reserved for de Rham differential *along* M. In (2.7) E denotes the momentum conjugate to A, the $d-p$-form. The (2.6) corresponds to the Hamiltonian vector field on $T^*\mathcal{A}_{\mathbb{R}}$, which, being a lift of a vector field on the configuration space $\mathcal{A}_{\mathbb{R}}$, is generated by the moment map, linear in E:

$$\mu = dE \tag{2.8}$$

which is traditionally called *Gauss law* in the context of gauge theory. Setting $\mu = 0$ and dividing by (2.6) defines the phase space of *abelian pure* *gauge p-form theory.* Hodge theory allows one to describe the quotient as $T^*\mathcal{V}$ where $\mathcal{V}$ is the vector space

$$\mathcal{V} = H^p(M,\mathbb{R}) \oplus (\mathrm{im} d^* \cap \Omega^p(M)) \tag{2.9}$$

while its dual $\mathcal{V}^*$ is identified with

$$\mathcal{V}^* = H^{d-p}(M,\mathbb{R}) \oplus \Big(\mathrm{im} d \cap \Omega^{d-p}(M)\Big) \tag{2.10}$$

Now that we have identified the phase space, let us look at the time evolution(s). The standard Hamiltonian of Maxwell theory is

$$\mathcal{H} = \int_M \frac{g^2}{2} E \wedge \star E + \frac{1}{2g^2} F \wedge \star F \tag{2.11}$$

with *curvature* $F = dA$, and some parameter g, called the gauge coupling.

If we use the spectral theory of the Laplacian

$$\Delta^{(p)} = d^* d + d d^* |_{\Omega^p(M)} \tag{2.12}$$

we can recognize in (2.11) an infinite-dimensional version of the system of *harmonic oscillators*, coupled to a *geodesic flow* on $H^p(M)$. The actual value of the coupling g is irrelevant, as we can change it by performing the canonical transformation, generated by

$$D = \int_M E \wedge A \tag{2.13}$$

☐ Why is D well-defined on $\mathcal{P}$? ■

2.1.1. *First glimpses of the ϑ-angle.* When $d = 2p+1$ the symplectic form (2.7) can be generalized to the family of forms:

$$\Omega_\vartheta = \int_M \delta E \wedge \delta A + \vartheta \int_M d\delta A \wedge \delta A \tag{2.14}$$

For odd p and $d = 2p$ the symplectic form (2.7) can be generalized to the family of forms:

$$\Omega_k = \int_M \delta E \wedge \delta A + k \int_M \delta A \wedge \delta A \tag{2.15}$$

Both generalizations correspond to the *magnetic field-rotating frame* generalization (1.33).

2.1.2. *Observables.* What are the natural observables in such a theory? The *electric field* E is gauge invariant, albeit constrained by the Gauss law. A closed $d-p$ form is characterized by its integrals over $d-p$-chains, measuring *electric fluxes*

$$\text{electric flux through } \Sigma_{d-p} = \int_{\Sigma_{d-p}} E \tag{2.16}$$

which does not change under the variations of the $d-p$-chain preserving its boundary:

$$\text{electric flux through } \Sigma_{d-p} = \text{electric flux through } \Sigma'_{d-p} \tag{2.17}$$

if

$$\Sigma_{d-p} - \Sigma'_{d-p} = \partial B_{d-p+1} \tag{2.18}$$

for some $d-p+1$-chain B_{d-p+1}. The observables can also be constructed out of A:

$$\text{generalized Bohm} - \text{Aharonov phase along } C_p = \int_{C_p} A \tag{2.19}$$

which is only gauge invariant for closed p-chains, $\partial C_p = 0$. The infinitesimal version of (2.19) is any functional of the curvature $F = dA$.

2.1.3. *Noncompact duality.* Suppose $d = 2p+1$. Remark that in this case the curvature $F = dA$ and the electric field are both $p+1$-forms. Also, both A and $\star E$ are p-forms.

2.2. **Compact p-forms.** An important generalization of the construction above cures the problem of continuous spectrum (infinite motion in the classical parlance) of the *zero mode sector* evolution above.

Let $\mathbf{t} \approx \mathbb{R}^r$ be a vector space, $\Gamma \subset \mathbf{t}$ a lattice $\approx \mathbb{Z}^r$ and $\mathbf{T} = \mathbf{t}/\Gamma$ the corresponding compact torus. The *compact p-form electrodynamics* is defined on

$$\mathcal{P} = T^* \mathcal{A}_{\mathbf{T}} // \mathcal{G}_{\mathbf{T}} \tag{2.20}$$

where the space $\mathcal{A}_{\mathbf{T}}$ is the space of *connections on the* $p-1$*-gerbe*, defined as follows. Pick a nice ${}^{\vee}C$ech covering $M = \cup_{\alpha\in A} U_\alpha$

$$\mathcal{A}_{\mathbf{T}} = \{ \, (A_\alpha) \mid A_\alpha \in \Omega^p(U_\alpha) \otimes \mathbf{t} \, , \qquad A_\alpha - A_\beta \in \Omega^p_\Gamma(U_\alpha \cap U_\beta) \, \} \tag{2.21}$$

where $\Omega^p_\Gamma(S) \subset \Omega^p(S) \otimes \mathbf{t}$ consists of all $\mathbf{t}$-valued p-forms $\mathfrak{o}$, *curvatures of* $p-2$*-gerbes defined on* $U_\alpha \cap U_\beta$, such that

$$\int_\Sigma \mathfrak{o} \in \Gamma \tag{2.22}$$

for any integral closed chain $\Sigma \in Z_p(S;\mathbb{Z})$. Note that the differential $F := dA$ is a globally defined $\mathbf{t}$-valued $p+1$-form, with Γ-valued periods:

$$\int_\Xi F \in \Gamma \tag{2.23}$$

for any integral closed chain $\Xi \in Z_{p+1}(S;\mathbb{Z})$. Indeed, the *overlap condition* implies $dA_\alpha = dA_\beta$ on any $U_\alpha \cap U_\beta$, thus F is globally well-defined. On the other hand

$$\int_\Xi F = \sum_\alpha \int_{\Xi\cap U_\alpha} F - \sum_{\alpha,\beta} \int_{\Xi\cap U_\alpha \cap U_\beta} F + \ldots = \tag{2.24}$$
$$\sum_\alpha \int_{\partial(\Xi\cap U_\alpha)} A_\alpha - \sum_{\alpha,\beta} \int_{\partial\left(\Xi\cap U_\alpha\cap U_\beta\right)} A_{\alpha\, or\ \beta} + \ldots$$

which may be nontrivial yet valued in Γ $\square$ Finish the argument $\blacksquare$.

The symplectic form is

$$\Omega_{T^*\mathcal{A}_{\mathbf{T}}} = \int_M \langle \delta E, \wedge \delta A \rangle \tag{2.25}$$

where $E \in \Omega^{d-p}(M) \otimes \mathbf{t}^*$ and we denote by $\langle \cdot, \cdot \rangle$ the pairing $\mathbf{t}^* \otimes \mathbf{t} \to \mathbb{R}$.

The gauge group $\mathcal{G}_{\mathbf{T}} = \Omega^p_{\Gamma}(M)$ acts by

$$(A_\alpha) \sim (A_\alpha + \varpi|_{U_\alpha}) \ , \ \varpi \in \Omega^p_{\Gamma}(M) \tag{2.26}$$

Any element $\varpi \in \Omega^p_{\Gamma}(M)$ defines a cohomology class, *a p-winding number*

$$[\varpi] \in H^p(M, \Gamma) \tag{2.27}$$

Any two ϖ', ϖ'' with the same p-winding number differ by an exact p-form

$$\varpi' - \varpi'' = d\xi \ , \ \xi \in \Omega^{p-1}(M) \tag{2.28}$$

The group $\mathcal{G}_{\mathbf{T}}$, therefore, is a direct product of a lattice $H^p(M, \Gamma)$ and a vector space $\Omega^{p-1}(M)/Z^{p-1}(M)$. The action of $\mathcal{G}_{\mathbf{T}}$ is the combination of the Hamiltonian vector fields generated by

$$\mu = dE \in \Omega^{d+1-p}(M) \otimes \mathbf{t}^* = (Lie \mathcal{G}_{\mathbf{T}})^* \tag{2.29}$$

and the action of the lattice $H^p(M, \Gamma)$ by shifts in $H^p(M, \mathbf{t})$. The latter is the orthogonal summand in the Hodge decomposition

$$\mathcal{A}_{\mathbf{T}} = \amalg_{H^{p+1}(M,\Gamma)} \ H^p(M, \mathbf{t}) \oplus (\mathrm{im} d^* \cap \Omega^p(M) \otimes \mathbf{t}) \tag{2.30}$$

2.2.1. *Observables in compact theory.* We can still define the electric fluxes, except that now they take values in $\mathbf{t}^*$, not in numbers, so we pair it with an element $\mu \in \mathbf{t}$ to land back in $\mathbb{R}$:

$$F_\mu(\Sigma_{(d-p)}) = \int_{\Sigma_{(d-p)}} \langle E, \mu \rangle \tag{2.31}$$

for a class of $d - p$-chains, defined up to a boundary $\Sigma_{(d-p)} \sim \Sigma_{(d-p)} + \partial B_{(d-p+1)}$. The generalized Bohm-Aharonov phase now becomes the generalized holonomy:

$$W_\lambda(C_{(p)}) = \exp 2\pi \mathrm{i} \int_{C_{(p)}} \langle \lambda, A \rangle \tag{2.32}$$

for a closed p-chain $C_{(p)}$, $\partial C_{(p)} = 0$.

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