1. Review of classical mechanics

The arena of most (but not all) of Classical mechanics is the world of ♠ symplectic manifolds \((\mathcal{P}, \omega)\), where \(\omega \in \Omega^2(\mathcal{P})\) is a closed, non-degenerate: 
\[
\omega \wedge^m \neq 0,
\]
two-form on a smooth manifold \(\mathcal{P}\) of dimension 
\[
\dim \mathcal{P} = 2m
\]  
\[\text{(1.1)}\]
supplemented by a choice of the Hamiltonian \(H \in C^\infty(\mathcal{P})\) ♦.
\[\text{(1.2)}\]
\[\mathcal{P}, \omega, H\]
with ,
The symplectic form makes the ring \(A = C^\infty(\mathcal{P})\) of smooth functions a Lie algebra, with the ♠ Poisson bracket given by 
\[
\{f, g\} = \iota\pi(df \wedge dg),
\]
with the Poisson bi-vector \(\pi = \omega^{-1}\).

□ The closedness \(d\omega = 0\) implies the Jacobi identity for \(\{\cdot, \cdot\}\):
\[
\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0
\]  
\[\text{(1.4)}\]
Examples:

(1) \(\mathcal{P} = T^*X\) where \(X\) is any manifold,
\[
\omega = d\theta
\]
with \(\theta \in \Omega^1(T^*X)\)-canonical 1-form
\[
\theta_{(p,q)}(\xi) = p(\pi_\ast \xi)
\]
\[\text{(1.6)}\]
where \(q \in X\), \(p \in T^*_qX\), \(\xi \in T_{(p,q)}T^*X\), \(\pi : T^*X \to X\) is the projection, and \(\pi_\ast \xi \in T_qX\) is the projection of the vector tangent to \(T^*X\) to the base tangent vector.

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Let \((M,\omega_M)\) be a symplectic manifold. Then \((\mathcal{P} = T^*M,\omega_\mathcal{P} = d\theta + k\pi^*\omega_M)\) defines a family of symplectic manifolds. The corresponding evolution is sometimes called a motion in magnetic field.

Let \(G\) be a simple Lie group, \(\mathfrak{g} = \text{Lie}G\) its Lie algebra, and \(\mathfrak{g}^*\) the dual space. Let \(\xi \in \mathfrak{g}\). Define \(P = O_\xi := \{\text{Ad}_g^*\xi | g \in G\}\) be the coadjoint orbit (of \(\xi\)). It carries the canonical Kirillov-Kostant symplectic form. Let us define it through the Poisson brackets of functions on \(\mathcal{P}\):

\[
\{f_1, f_2\}(x) = x([df_1, df_2])
\]

Here the functions \(f_{1,2} : \mathfrak{g}^* \to \mathbb{R}\) have differentials \(df_{1,2}\) which, at the point \(x \in \mathcal{P} \subset \mathfrak{g}^*\) are the linear functions on \(T_x\mathcal{P} \approx \mathfrak{g}^*\), i.e. (for finite dimensional vector spaces \(V \approx V^{**}\)) elements \(v_{1,2} \in \mathfrak{g}\). We then evaluate \(x\), as a linear function on \(\mathfrak{g}\), on the commutator \([v_1, v_2]\). □ Show (1.7) is invertible i.e. corresponds to a symplectic form.

Specifically, let \(G = SU(N)\). We can view \(G\) as a subgroup of the group \(U(N)\) of automorphisms of the \(N\)-dimensional vector space \(N \approx \mathbb{C}^N\) endowed with a hermitian form, i.e. sesquilinear non-degenerate pairing \(\langle \cdot, \cdot \rangle : N \times N \to \mathbb{C}\), obeying

\[
\langle xv_1, yv_2 \rangle = \bar{x}y\langle v_1, v_2 \rangle, \quad \langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}
\]

for any \(x, y \in \mathbb{C}\), \(v_1, v_2 \in V\). So, \(g \in \text{GL}(\mathbb{N})\) belongs to \(U(N)\) if for any \(v_1, v_2 \in V\)

\[
\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle
\]

The subgroup \(G\) is singled out by the condition \(\det(g) = 1\). In other words, special unitary transformations preserve a volume form \(\Omega \in \Lambda^N\mathbb{N}^*\), in addition to the hermitian form. Now recall that the operator \(A \in \text{End}(V)\) is called hermitian, if for any \(v_1, v_2 \in V\)

\[
\langle v_1, Av_2 \rangle = \langle Av_1, v_2 \rangle
\]

Let us choose a basis \(e_1, \ldots, e_N\), orthonormal with respect to the hermitian form:

\[
\langle e_i, e_j \rangle = \delta_{ij}
\]

In this basis the operators \(g, A\) have the associated matrices \(\|g_{ij}\|, \|A_{ij}\|\),

\[
g_{ij} = \langle e_i, ge_j \rangle, \quad A_{ij} = \langle e_i, Ae_j \rangle
\]

(the bar on \(\bar{j}\) signifies the different role \(e_i\) and \(e_j\) play in the right hand side of the equation).
Show the unitarity of $g$ and hermiticity of $A$ is equivalent to the set of equations

$$gg^\dagger = 1_N \iff \sum_{j=1}^N g_{ij}g_{jk} = \delta_{ik}, \quad i, k = 1, \ldots, N$$

$$A = A^\dagger \iff A_{ij} = A_{ji}$$

The Lie algebra $\mathfrak{g} = \text{Lie} U(N)$ is the vector space of all anti-hermitian operators in $\mathbb{N}$:

$$B \in \mathfrak{g} \iff \langle v_1, Bv_2 \rangle + \langle Bv_1, v_2 \rangle = 0$$

Of course, if $B$ is antihermitian, then $A = iB$ is hermitian and vice versa. The Lie algebra of $SU(N)$ is a subspace of all traceless antihermitian matrices. Consider the set of all hermitian operators with fixed eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$:

$$\mathcal{O}_{\lambda_1, \ldots, \lambda_N} = \left\{ A \mid \text{Det}(\lambda - A) = \prod_{i=1}^N (\lambda - \lambda_i) \right\}$$

Using the pairing

$$\langle A, B \rangle := \text{itr}_N AB$$

we identify $\mathfrak{g}^*$ with the space of Hermitian operators in $\mathbb{N}$. Thus, $\mathcal{O}_{\lambda_1, \ldots, \lambda_N} \subset \mathfrak{g}^*$ is a coadjoint orbit of $U(N)$. To make it into the coadjoint orbit of $SU(N)$ we need to descend to the quotient of the space of all Hermitian matrices by the action of $\mathbb{R}$ of shifts by a scalar operator:

$$B \sim B + b \cdot 1_N, \quad b \in \mathbb{R}$$

We can fix the representative by demanding that the $B$ operators are also tracefree, $\text{tr}B = 0$. Thus, let

$$\lambda_1 + \ldots + \lambda_N = 0$$

Then, $\mathcal{O}_{\lambda_1, \ldots, \lambda_N} \subset \mathfrak{su}(N)^*$ is a coadjoint orbit of $SU(N)$. Flag varieties, Grassmanians, projective spaces.

As is customary in theoretical physics we shall take the above definitions and try our best in extending them to the infinite-dimensional settings. The loop space $LX = \text{Maps}(S^1, X)$ of smooth maps of a circle to a Riemannian manifold $(X, g_X)$ carries a closed two-form $\Omega_{LX}$. At some loop $\gamma \in LX$ its value on a pair of vectors $\xi_1, \xi_2 \in \Gamma(S^1, \gamma^*TX)$ is given by:

$$\Omega_{LX}(\xi_1, \xi_2) = \int_{S^1} \gamma^* g_X(\xi_1, \nabla \xi_2)$$

where $\nabla$ is the pull-back by $\gamma$ of the Levi-Civita connection on $TX$ defined by the metric $g_X$. □ Is $\Omega_{LX}$ a symplectic form? □.
(6) This example can be generalized to the case of $P = \text{Maps}(M, X)$, where $M$ is a compact manifold of dimension $n$, endowed with a closed $n-1$-form $\nu_M$. We define

\begin{equation}
\Omega^X_M(\xi_1, \xi_2) = \int_M \nu_M \wedge \gamma^*g_X(\xi_1, \nabla\xi_2)
\end{equation}

(7) Let $(M, \mu_M)$ be a compact manifold endowed with a volume form $\mu_M \in \Omega^{\dim(M)}(M)$, $\mu_M \neq 0$, and let $(X, \omega_X)$ be a symplectic manifold. Define $P = \text{Maps}(M, X)$, and endow it with the symplectic form, s.t. at $\gamma : M \to X$ and $\xi_1, \xi_2 \in \Gamma(M, \gamma^*TX)$

\begin{equation}
\Omega^X_M(\xi_1, \xi_2) = \int_M \mu_M \gamma^*\omega_X(\xi_1, \xi_2)
\end{equation}

1.1. Hamilton equations. Now let us put the function $H \in A$ to a good use. The differential $dH$ is a $1$-form on $P$. Define the Hamiltonian vector field $V_H$ by:

\begin{equation}
\iota_{V_H}\omega = dH \iff V_H = \iota_{\omega^{-1}}dH
\end{equation}

Examples:

(1) Let $X$ be any manifold and $v \in \text{Vect}(X)$ a vector field. Let $P = T^*X$ with $\omega = d\theta$, and $H(p, q) = p(v(q))$, $q \in X$, $p \in T^*_qX$. The corresponding vector field $V_H$ covers the vector field $v$ on $X$. □

Compute $V_H$. ■

(2) Let $(X, g)$ be a Riemannian manifold. We view the metric $g$ as a smooth map $TX \to T^*X$. Then $(P = T^*X, \omega = d\theta, H)$ with

\begin{equation}
H = \frac{1}{2}(p, g^{-1}p)
\end{equation}

defines the Hamiltonian system, covering the geodesic flow on $X$.  

(3) Again, let $(X, g)$ be a Riemannian manifold and $U \in C^\infty(X)$. Then

\begin{equation}
H(p, q) = \frac{1}{2}(p, g(q)^{-1}p) + U(q)
\end{equation}

describes a particle on $X$ moving in the field of the potential $U$.  

\textsuperscript{1} ♠ A geodesic curve on a Riemannian manifold $(X, g)$ is the extremum of the functional

\begin{equation}
L = \int_U \sqrt{g(\dot{\ell}, \dot{\ell})} \, dt
\end{equation}

on the space $\text{Maps}(U, X)/\text{Diff}_+(U)$ of oriented curves in $X$. Here $U \subset \mathbb{R}_t$ is a connected domain, and $\ell : U \to X$ a smooth map. One can impose Dirichlet boundary conditions: $\ell(\partial U) = \{x_s, x_t\}$ with two points $x_s, x_t \in X$. Analogously, a minimal surface in $X$ is the extremum of the functional

\begin{equation}
A = \int_U \sqrt{g(\ell', \ell')}g(\ell', \ell') - g(\ell', \ell')^2 \, dt ds
\end{equation}

on the space $\text{Maps}(U, X)/\text{Diff}_+(U)$ of oriented surfaces in $X$. Here $U \subset \mathbb{R}^{2, s}_t$ is a connected domain, and $\ell : U \to X$ a smooth map. One can impose Dirichlet boundary conditions: $\ell(\partial U) = \gamma$, $\gamma \subset X$, or Neumann boundary conditions: $\nabla_n \ell |_{\partial U} = 0$. ◇
(4) An important special case of the system above is the generalized harmonic oscillator (or a system of \(m\) oscillators): \(X = T^*Q, \ Q = \mathbb{R}^m, \ \omega = \sum_{i=1}^m dp_i \wedge dq_i\),

\[
H = \frac{1}{2} \sum_{i,j=1}^m g^{ij} p_i p_j + \sum_{i,j=1}^m K^i_j p_j q^i + \frac{1}{2} \sum_{i,j=1}^m U_{ij} q^i q^j
\]

where we assume the non-degenerate positive definite metric \(g_{ij}\) on \(Q\), whose inverse \(g^{ij}\) appears as the kinetic term in (1.27). We also assume the second metric \(U_{ij}\) on the same \(Q\), defining the potential term. The cross-term \((p, Kq)\) depends on a choice of linear operator \(K: V \to V\). The operator \(K\) can be decomposed as a sum of \(g\)-symmetric and \(g\)-antisymmetric parts:

\[
K = K^s + K^a, \ gK^s = (K^s)^t g, \ gK^a = -(K^a)^t g
\]

where we viewed the metric \(g\) as the symmetric map \(g: V \to V^*\), \(2g^t = g\). The symmetric part \(K^s\) can be eliminated from \(H\) by a canonical transformation:

\[
p \mapsto p - g(K^s q)
\]

\[
\sum d \left( g_{il} K^l_m q^m \right) \wedge dq_i = 0
\]

What is the meaning of \(K^a\)? Define \(\Omega = gK^a, \ \Omega^t = -\Omega\). Then we can re-write the Hamiltonian as:

\[
H = \frac{1}{2} g^{ij} \left( p_i + \Omega_{ik} q^k \right) \left( p_j + \Omega_{jl} q^l \right) + \frac{1}{2} V_{ij} q^i q^j
\]

where \(V\) is related to \(U\) in an obvious way. Finally, we can reinterpret (1.32) as the Hamiltonian evolution of a system of oscillators with standard kinetic and potential terms

\[
\tilde{H} = \frac{1}{2} g^{ij} p_i p_j + \frac{1}{2} V_{ij} q^i q^j
\]

but with a modified symplectic form:

\[
\omega_{\text{mt}} = dp_i \wedge dq_i + \frac{1}{2} \Omega_{ij} dq_i dq^j
\]

This modification is sometimes called magnetic field-rotating frame.

\[\square\] Why? \[\blacksquare\].

(5) Now let us assume \(X\) is endowed with a closed two-form \(F\). Let \(\mathcal{P} = T^*X\) with \(\omega = d\theta + \pi^* F\). \(\square\) Compute \(V_H\) for \(H\) given by the Eq. (1.26). \(\blacksquare\).

\[\diamondsuit\] For a linear map \(A: V_1 \to V_2\) we denote by \(A^t\) the canonical dual map \(V_2^* \to V_1^*\), defined by \(A^t(\xi)(v) = \xi(Av)\), for any \(v \in V_1, \xi \in V_2^*\).

\[\checkmark\] Throughout the notes we adopt Einstein’s convention except where it leads to confusion: \(A^{\ldots}_{i\ldots}B_{i\ldots}^{\ldots} := \sum_i A^{\ldots}_{i\ldots}B_{i\ldots}^{\ldots}\).
1.2. **Darboux coordinates, action-angle variables.** The local model of \( P \) is a domain in \( T^\ast \mathbb{R}^m \approx \mathbb{R}^{2m} \), with \( \omega = d\theta, \theta = \sum_{i=1}^m p_i dq^i \) with \( (q^i) \) coordinates on \( \mathbb{R}^m \). We can change the coordinates by symplectic (canonical) diffeomorphisms. Let \( g : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) preserves \( \omega \), then

\[
g^\ast \theta - \theta = dS
\]

for some function \( S \). In coordinates:

\[
\sum_{i=1}^m P_i dQ^i - p_i dq^i = dS
\]

thus a function of \( m + m \) variables old and new \( q \)'s \( S = S(Q,q) \) is all is needed to generate the symplectomorphism \( g \) (a beautifully non-invariant yet useful formalism):

\[
P_i = \frac{\partial S}{\partial Q^i}, \quad p_i = -\frac{\partial S}{\partial q^i}
\]

Examples:

1. Linear symplectomorphisms are generated by the quadratic functions

\[
S = \sum_{i,j=1}^m \frac{1}{2} A_{ij} Q^i Q^j + B_{ij} Q^i q^j + \frac{1}{2} C_{ij} q^i q^j
\]

2. In particular, time evolution of a system of harmonic oscillators is described by:

3. Change of cartesian to polar coordinates on \( \mathbb{R}^2 \):

\[
(p, q) \mapsto (A, \varphi) \quad A = \frac{p^2 + q^2}{2}, \quad q = \sqrt{2A} \cos(\varphi), \quad p = \sqrt{2A} \sin(\varphi)
\]

□ What is its generating function \( S(\varphi, q) \)? ■

1.3. **Symplectic quotients.** Suppose \( (P, \omega, H) \) is invariant under the action of a Lie group \( G \). Moreover, we’ll require the action of \( G \) to be Hamiltonian (this is automatic for simply-connected \( P \) why? ■).

A diffeomorphism \( g : P \to P \) is a symplectomorphism if it preserves \( \omega \):

\[
g^\ast \omega = \omega
\]

Infinitesimal symplectomorphism is a vector field \( V \in \text{Vect}(P) \), such that

\[
\text{Lie}_V \omega = 0 \iff d(\iota_V \omega) = 0
\]

A Hamiltonian \( G \)-action on \( P \) associates to every \( \xi \in g \), a Hamiltonian vector field \( V_\xi \),

\[
\iota_{V_\xi} \omega = dh_\xi
\]

with some Hamiltonian function \( h_\xi : P \to \mathbb{R} \). Of course (1.41) defines \( h_\xi \) up to a constant. We can partly restrict the choice of a constant by requiring the map \( \xi \mapsto h_\xi \) be linear. We have the homomorphism condition

\[
[V_\xi, V_\zeta] = V_{[\xi, \zeta]}
\]
for any $\xi_1, \xi_2 \in \mathfrak{g}$. We may want to require that
\begin{equation}
(1.43) \quad h_{[\xi_1, \xi_2]} = \{h_{\xi_1}, h_{\xi_2}\} := \omega(V_{\xi_1}, V_{\xi_2}) = \iota_{\omega^{-1}} dh_{\xi_1} \wedge dh_{\xi_2}
\end{equation}
This is not always possible. For example, the abelian group $\mathbb{R}^2$ of translations acts on $\mathbb{R}^2$ preserving the constant volume 2-form $dp \wedge dx$. The Hamiltonians $h_1, h_2$ are equal to $p, x$, respectively. The corresponding vector fields commute, however $p, x = 1 \neq 0$.

Algebraically, the problem of finding the constants $c_\xi$ adjusting $h_\xi$ so as to obey (1.43) is the question of whether the Lie algebra $\mathfrak{g}$ has non-trivial second cohomology. Define
\begin{equation}
(1.44) \quad c(\xi_1, \xi_2) = \{h_{\xi_1}, h_{\xi_2}\} - h_{[\xi_1, \xi_2]}
\end{equation}
As it stands $c(\xi_1, \xi_2)$ is a function on $\mathcal{P}$. However,
\begin{equation}
(1.45) \quad dc(\xi_1, \xi_2) = \iota_{[V_{\xi_1}, V_{\xi_2}]} \omega - \iota_{V_{[\xi_1, \xi_2]}} \omega = 0
\end{equation}
by Eq. (1.42). Thus, $c : \Lambda^2 \mathfrak{g} \to \mathbb{R}$ defines a linear map. It obeys:
\begin{equation}
(1.46) \quad c([\xi_1, \xi_2], \xi_3) + c([\xi_2, \xi_3], \xi_1) + c([\xi_3, \xi_1], \xi_2) = \{h_{[\xi_1, \xi_2]}, h_{\xi_3}\} + \text{cyclic} = 0
\end{equation}
where we used the linearity of $\xi \mapsto h_\xi$ map and $\{h_{[\xi_1, \xi_2]}, \cdot\} = \{h_{\xi_1}, h_{\xi_2}\} + \text{cyclic}$.

1.3.1. Cohomology of groups and algebras. Let us pause to define an interesting cohomology theory. Let $\mathfrak{g}$ be a Lie algebra, and $M$ a $\mathfrak{g}$-module, i.e. a vector space with the homomorphism $\rho : \mathfrak{g} \to \text{End}(M)$. For $\xi \in \mathfrak{g}, m \in M$ we denote $\rho(\xi)m$ simply by $\xi \cdot m$. Define $C^i(\mathfrak{g}, M)$ to be the space of all skew-symmetric polylinear functions $c(i) : \Lambda^i \mathfrak{g} \to M$. Define the differential $\delta : C^i(\mathfrak{g}, M) \to C^{i+1}(\mathfrak{g}, M)$ by
\begin{equation}
(1.47) \quad \delta c(i)(\xi_1 \wedge \ldots \wedge \xi_{i+1}) = \sum_{j=1}^{i+1} (-1)^{j-1} \xi_j \cdot c(i) \left( \xi_1 \wedge \ldots \wedge \hat{\xi}_j \wedge \ldots \wedge \xi_{i+1} \right) + \sum_{1 \leq a < b \leq i+1} (-1)^{a+b} c(i) \left( [\xi_a, \xi_b] \wedge \xi_1 \wedge \ldots \wedge \hat{\xi}_a \wedge \ldots \wedge \wedge \xi_{i+1} \right)
\end{equation}
\[\Box\] Check, that $\Box$ it is a differential, $\delta^2 = 0$.

One defines $\blacklozenge$ the cohomology groups
\begin{equation}
(1.48) \quad H^i(\mathfrak{g}, M) = \left( \ker \delta \cap C^i \right) / \left( \text{im} \delta \cap C^i \right)
\end{equation}

Let us now take, as example, $M = \mathbb{R}$ with a trivial action of $\mathfrak{g}$. The $H^2(\mathfrak{g}, \mathbb{R})$ group is the space of all skew-symmetric bilinear forms $c(\xi \wedge \eta)$ on $\mathfrak{g}$ which are closed under the differential
\begin{equation}
(1.49) \quad \delta c(\xi_1 \wedge \xi_2 \wedge \xi_3) = c([\xi_1, \xi_2] \wedge \xi_3) + c([\xi_2, \xi_3] \wedge \xi_1) + c([\xi_3, \xi_1] \wedge \xi_2)
\end{equation}
modulo $\delta$-exact forms $c \sim c + \delta b$, where $\delta b(\xi \wedge \eta) = b([\xi, \eta])$. For simple Lie algebras $H^2$ vanishes. For abelian Lie algebra $\mathfrak{a}$ the space of $\mathbb{R}$-valued
$i$-cocycles is the space of all skew-symmetric $i$-linear functions on $a$, since
the commutators vanish implying the vanishing $\delta$. In other words,
\begin{equation}
H^i(a, \mathbb{R}) = \Lambda^i a^*
\end{equation}

Any 2-cocycle $c \in Z^2(g, \mathbb{R})$ defines a central extension of $g$, i.e. a new Lie
algebra $\tilde{g}$, which is equal to $g \oplus \mathbb{R}$ as a vector space, with the commutator
defined by:
\begin{equation}
[(\xi_1, c_1), (\xi_2, c_2)] = ([\xi_1, \xi_2], c(\xi_1 \wedge \xi_2))
\end{equation}
However, not all such extensions are non-trivial. Indeed, $\xi \mapsto (\xi, b(\xi))$
embeds $g$ into $\tilde{g}$ for linear function $b \in g^*$. Thus, only cocycles modulo
coboundaries produce non-trivial, new, Lie algebras.

The most basic such extension is, in fact, an avatar of quantization. Given
a symplectic vector space $(V, \omega)$ we define ♠ the Heisenberg algebra $H_V$
\begin{equation}
\text{to be the central extension of } V \text{ viewed as abelian Lie algebra, corresponding}
\end{equation}
\begin{equation}
to \omega \text{ viewed as the Lie algebra-cohomology class } \omega \in H^2(V, \mathbb{R}).
\end{equation}
1.3.2. Moment map. Given a symplectic manifold $(\mathcal{P}, \omega)$ and a Lie group $G$
acting on $\mathcal{P}$ by Hamiltonian vector fields, define
\begin{equation}
pred = \mathcal{P}^\mu / G = \mu^{-1}(\zeta)/G
\end{equation}
where, assuming the vanishing of the obstruction class in $H^2(g, \mathbb{R})$, the moment map
\begin{equation}
\mu : M \to g^*
\end{equation}
is defined via:
\begin{equation}
\langle \mu(x), \xi \rangle = h_\xi(x), \xi \in g, \; x \in M
\end{equation}
As discussed above, the possibility of adding a constant to $h_\xi$ translates to
a variety of possibilities for $\zeta \in (g^*)^G$ in (1.52). For simple Lie group $G$ only
$\zeta = 0$ is possible, while for $G$ with center there are many options for the
level of the moment map. For example, take $M = \mathbb{R}^{2m}$, $\omega = \sum_{i=1}^{m} dp_i \wedge dx_i$, $G = U(1)$ acting via:
\begin{equation}
e^{it} : (p, x) \mapsto (p \cos(t) - x \sin(t), \; x \cos(t) + p \sin(t))
\end{equation}
The moment map
\begin{equation}
\mu = \frac{1}{2} \sum_{i=1}^{m} (p_i^2 + (x_i)^2)
\end{equation}
The reduced phase space is empty for $\zeta < 0$, a point for $\zeta = 0$, and the
remarkably important compact symplectic manifold of dimension $2(m - 1)$
for $\zeta > 0$, the complex projective space $\mathbb{C}P^{m-1}$. As a bonus, it carries a
complex structure, and so it can be described in complex terms only:
\begin{equation}
\mathbb{C}P^{m-1} = \{(z_1, \ldots, z_m) | z \neq 0, \; z \sim uz, \; u \in \mathbb{C}^\times \} = (\mathbb{C}^m \setminus 0) / \mathbb{C}^\times
\end{equation}
In other words, we encountered the relation $M^\mu / G = M^s / G_C$ which will be
discussed in greater generality later.
2. Review of classical electromagnetism

Maxwell theory, or classical electromagnetism, is the infinite-dimensional version of classical mechanics, where the phase space $\mathcal{P}$ is the field space. We shall discuss it in somewhat artificial setting of compact spaces, which is easier to handle mathematically, and has applications to topology.

Let $M^d$ be a compact $d$-dimensional manifold, endowed with Riemannian metric $g$. The metric defines the Hodge star operator $\star$ mapping $p$-forms to $d-p$-forms via the point-wise relation

$$\omega \star \omega = \text{vol}_g \sum_{i_1 < i_2 < \ldots < i_p} \omega(e_{i_1}, \ldots, e_{i_p})\omega(f^{i_1}, \ldots, f^{i_p})$$

where $e_i, i = 1, \ldots, m$ is any basis in $T_xM$, and $f^i$ the associated orthogonal basis, such that

$$g(e_i, f^j) = \delta^j_i$$

for all $i, j = 1, \ldots, m$. The Eq. (2.1) endows $\Omega^p$ with the metric

$$(\omega, \eta) = \int_M \omega \wedge \star \eta = (\eta, \omega)$$

We shall glide over the finer details such as $L^2$ completions, Sobolev embeddings etc. which are needed to be able to operate with

$$d^* = \star d \star,$$

the operator $\Omega^i(M) \to \Omega^{i-1}(M)$, conjugate to $d$ in the sense of the Hodge metric (2.3) on the space of differential forms.

2.1. $\mathbb{R}$-gauge $p$-form theory. Our first approximation is to take

$$\mathcal{P} = T^*A_{\mathbb{R}}/S_{\mathbb{R}}$$

where $A_{\mathbb{R}} = \Omega^p(M)$, $S_{\mathbb{R}} = (\Omega^{p-1}(M)/Z^{p-1}(M))$ is the vector space of $p-1$-forms considered modulo closed ones, acting on $A_{\mathbb{R}}$ by

$$A \mapsto A + d\xi$$

for $A \in A_{\mathbb{R}}, \xi \in S_{\mathbb{R}}$. The symplectic form on $T^*A_{\mathbb{R}}$ is traditionally written as

$$\Omega_{T^*A_{\mathbb{R}}} = \int_M \delta E \wedge \delta A$$

where $\delta$ denotes the de Rham differential in the space of fields while $d$ is reserved for de Rham differential along $M$. In (2.7) $E$ denotes the momentum conjugate to $A$, the $d-p$-form. The (2.6) corresponds to the Hamiltonian vector field on $T^*A_{\mathbb{R}}$, which, being a lift of a vector field on the configuration space $A_{\mathbb{R}}$, is generated by the moment map, linear in $E$:

$$\mu = dE$$

which is traditionally called Gauss law in the context of gauge theory. Setting $\mu = 0$ and dividing by (2.6) defines the phase space of abelian pure
gauge $p$-form theory. Hodge theory allows one to describe the quotient as $T^*V$ where $V$ is the vector space

$$V = H^p(M, \mathbb{R}) \oplus (\text{im}d \cap \Omega^p(M))$$

while its dual $V^*$ is identified with

$$V^* = H^{d-p}(M, \mathbb{R}) \oplus \left(\text{im}d \cap \Omega^{d-p}(M)\right)$$

Now that we have identified the phase space, let us look at the time evolution(s). The standard Hamiltonian of Maxwell theory is

$$H = \int_M \frac{g^2}{2} E \wedge \star E + \frac{1}{2g^2} F \wedge \star F$$

with curvature $F = dA$, and some parameter $g$, called the gauge coupling.

If we use the spectral theory of the Laplacian

$$\Delta^{(p)} = d^*d + dd^*|_{\Omega^p(M)}$$

we can recognize in (2.11) an infinite-dimensional version of the system of harmonic oscillators, coupled to a geodesic flow on $H^p(M)$. The actual value of the coupling $g$ is irrelevant, as we can change it by performing the canonical transformation, generated by

$$D = \int_M E \wedge A$$

□ Why is $D$ well-defined on $\mathcal{P}$? ■

2.1.1. First glimpses of the $\vartheta$-angle. When $d = 2p + 1$ the symplectic form (2.7) can be generalized to the family of forms:

$$\Omega_\vartheta = \int_M \delta E \wedge \delta A + \vartheta \int_M d\delta A \wedge \delta A$$

For odd $p$ and $d = 2p$ the symplectic form (2.7) can be generalized to the family of forms:

$$\Omega_k = \int_M \delta E \wedge \delta A + k \int_M \delta A \wedge \delta A$$

Both generalizations correspond to the magnetic field-rotating frame generalization (1.33).

2.1.2. Observables. What are the natural observables in such a theory? The electric field $E$ is gauge invariant, albeit constrained by the Gauss law. A closed $d-p$ form is characterized by its integrals over $d-p$-chains, measuring electric fluxes

$$\text{electric flux through } \Sigma_{d-p} = \int_{\Sigma_{d-p}} E$$

which does not change under the variations of the $d-p$-chain preserving its boundary:

$$\text{electric flux through } \Sigma_{d-p} = \text{electric flux through } \Sigma'_{d-p}$$
if
\[ (2.18) \Sigma_{d-p} - \Sigma'_{d-p} = \partial B_{d-p+1} \]
for some \( d-p+1 \)-chain \( B_{d-p+1} \). The observables can also be constructed out of \( A \):
\[ (2.19) \text{generalized Bohm–Aharonov phase along } C_p = \int_{C_p} A \]
which is only gauge invariant for closed \( p \)-chains, \( \partial C_p = 0 \). The infinitesimal version of (2.19) is any functional of the curvature \( F = dA \).

2.1.3. Noncompact duality. Suppose \( d = 2p+1 \). Remark that in this case the curvature \( F = dA \) and the electric field are both \( p+1 \)-forms. Also, both \( A \) and \( \ast E \) are \( p \)-forms.

2.2. Compact \( p \)-forms. An important generalization of the construction above cures the problem of continuous spectrum (infinite motion in the classical parlance) of the zero mode sector evolution above.

Let \( t \approx \mathbb{R}^r \) be a vector space, \( \Gamma \subset t \) a lattice \( \approx \mathbb{Z}^r \) and \( T = t/\Gamma \) the corresponding compact torus. The compact \( p \)-form electrodynamics is defined on
\[ (2.20) \mathcal{P} = T^\ast \mathcal{A}_T/\mathcal{G}_T \]
where the space \( \mathcal{A}_T \) is the space of connections on the \( p-1 \)-gerbe, defined as follows. Pick a nice \( \vee \)Čech covering \( M = \cup_{\alpha \in A} U_\alpha \)
\[ (2.21) \mathcal{A}_T = \{ (A_\alpha) \mid A_\alpha \in \Omega^p(U_\alpha) \otimes t, \quad A_\alpha - A_\beta \in \Omega^p(U_\alpha \cap U_\beta) \} \]
where \( \Omega^p(U) \subset \Omega^p(S) \otimes t \) consists of all \( t \)-valued \( p \)-forms \( o \), \( \text{curvatures of } p-2 \text{-gerbes defined on } U_\alpha \cap U_\beta \), such that
\[ (2.22) \int_\Sigma o \in \Gamma \]
for any integral closed chain \( \Sigma \in Z_p(S; \mathbb{Z}) \). Note that the differential \( F := dA \) is a globally defined \( t \)-valued \( p+1 \)-form, with \( \Gamma \)-valued periods:
\[ (2.23) \int_\Xi F \in \Gamma \]
for any integral closed chain \( \Xi \in Z_{p+1}(S; \mathbb{Z}) \). Indeed, the overlap condition implies \( dA_\alpha = dA_\beta \) on any \( U_\alpha \cap U_\beta \), thus \( F \) is globally well-defined. On the other hand
\[ (2.24) \int_\Xi F = \sum_\alpha \int_{\Xi \cap U_\alpha} F - \sum_{\alpha,\beta} \int_{\Xi \cap U_\alpha \cap U_\beta} F + \ldots = \sum_\alpha \int_{\partial(\Xi \cap U_\alpha)} A_\alpha - \sum_{\alpha,\beta} \int_{\partial(\Xi \cap U_\alpha \cap U_\beta)} A_\alpha \ast \beta + \ldots \]
which may be nontrivial yet valued in \( \Gamma \) ■ Finish the argument ■.
The symplectic form is
\[ \Omega^{\ast, A_T} = \int_M \langle \delta E, \wedge \delta A \rangle \]
where \( E \in \Omega^{d-p}(M) \otimes \mathfrak{t}^* \) and we denote by \( \langle \cdot, \cdot \rangle \) the pairing \( \mathfrak{t}^* \otimes \mathfrak{t} \to \mathbb{R} \).

The gauge group \( G_T = \Omega^p_t(M) \) acts by
\[ (A_\alpha) \sim (A_\alpha + \varpi | U_\alpha), \quad \varpi \in \Omega^p_t(M) \]
Any element \( \varpi \in \Omega^p_t(M) \) defines a cohomology class, a \( p \)-winding number
\[ [\varpi] \in H^p(M, \Gamma) \]
Any two \( \varpi', \varpi'' \) with the same \( p \)-winding number differ by an exact \( p \)-form
\[ \varpi' - \varpi'' = d\xi, \quad \xi \in \Omega^{p-1}(M) \]
The group \( G_T \), therefore, is a direct product of a lattice \( H^p_t(M, \Gamma) \) and a vector space \( \Omega^{p-1}(M)/Z^{p-1}(M) \). The action of \( G_T \) is the combination of the Hamiltonian vector fields generated by
\[ \mu = dE \in \Omega^{d+1-p}(M) \otimes \mathfrak{t}^* = (\text{Lie } G_T)^* \]
and the action of the lattice \( H^p(M, \Gamma) \) by shifts in \( H^p(M, \mathfrak{t}) \). The latter is the orthogonal summand in the Hodge decomposition
\[ A_T = \Pi_{H^{p+1}(M, \Gamma)} H^p(M, \mathfrak{t}) \oplus (\text{im} d^* \cap \Omega^p(M) \otimes \mathfrak{t}) \]
2.2.1. Observables in compact theory. We can still define the electric fluxes, except that now they take values in \( \mathfrak{t}^* \), not in numbers, so we pair it with an element \( \mu \in \mathfrak{t} \) to land back in \( \mathbb{R} \):
\[ F_\mu(\Sigma_{(d-p)}) = \int_{\Sigma_{(d-p)}} \langle E, \mu \rangle \]
for a class of \( d-p \)-chains, defined up to a boundary \( \Sigma_{(d-p)} \sim \Sigma_{(d-p)} + \partial B_{(d-p+1)} \). The generalized Bohm-Aharonov phase now becomes the generalized holonomy:
\[ W_\lambda(C_{(p)}) = \exp 2\pi i \int_{C_{(p)}} \langle \lambda, A \rangle \]
for a closed \( p \)-chain \( C_{(p)} \), \( \partial C_{(p)} = 0 \).

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