WINDING NUMBERS AND THE BORSUK-ULAM THEOREMS.

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1 Winding Number

One can motivate our investigation of winding numbers by considering induced maps on Homology. It is useful to think of this mapping as akin to a winding or linking type construction. Consider a mapping \( f : X \to Y \), a natural question when considering Homology would then be, to what extent are the holes of \( X \) wound about the holes of \( Y \) given this mapping.

Let \( x \in \mathbb{R}^2 \) and let \( \gamma \) be an oriented close curved defined on \( \mathbb{R}^2 - \{x\} \), we can then intuitively define the winding number \( w(x, \gamma) \) as:

\[
w(x, \gamma) = \text{"the number of times } \gamma \text{ winds about } x,"
\]

where the winding number is positive for counterclockwise rotations about a hole, and negative for clockwise rotations.

Now that we’ve laid out some basic intuition for winding numbers, let’s make our definition rigorous. A good, simple starting place for discussing winding numbers is dealing with the plane \( \mathbb{R}^2 \) and from there we will move to define the winding number in a more general topological setting.

1.1 Angles in the Plane

When considering an angle formed by a vector in the plane, we must consider that it is not single valued, but rather an element \([\phi] = \{\phi + 2\pi n : n \in \mathbb{Z}\}\) of the quotient set \( \mathbb{R}/2\pi\mathbb{R} \).

For example: Consider \( \vec{a} = (1, 1) \in \mathbb{R}^2 \), then \( \text{arg}(\vec{a}) = \frac{\pi}{4} + 2\pi n, n \in \mathbb{Z} = [\frac{\pi}{4}] \). We should also note that \( \mathbb{R}/2\pi\mathbb{R} \) is an abelian group. To avoid this we will generally restrict \( \text{arg} \), to \( \bar{\text{arg}} \) where \( \phi \) only takes values on \( (-\pi, \pi) \).

1.2 The Angle Subtended by a Line Segment

If we let \( a, b \) be two points in the plane \( \mathbb{R}^2 \), then we denote \( [a, b] \) as the directed line segment from \( a \) to \( b \). We paramaterize this as:

\[
\gamma : [0, 1] \to \mathbb{R}^2; \gamma(t) = a + t(b - a)
\]

and we define the support of \( [a, b] \) as \( [a, b] \) where

\[
|[a, b]| = \{a + t(b - a); t \in [0, 1]\}
\]

or simply the collection of points on our line segment.

**Definition (Angle Cocycle).** Let \( a, b \in \mathbb{R}^2 \) and suppose that \( 0 \not\in [a, b] \), then we define \( \theta([a, b], 0) \) as the unique \( \theta \in (-\pi, \pi) \) such that \( [\theta] = \text{arg}(b) - \text{arg}(a) \). More generally we can define \( \theta([a, b], x) \) as \( \theta([a - x, b - x], 0) \). This angle cocycle function is continuous over its domain \( \mathbb{R}^2 - [a, b] \).

1.3 Discrete Winding Number

**Definition (Directed Polygon and its Support).** Let \( \gamma = P(a_0, a_1 \ldots a_n) \) be the directed polygon whose vertices are \( a_0, a_1 \ldots a_n \in \mathbb{R}^2 \) whose edges are the directed line segments \( [a_0, a_1], [a_1, a_2] \ldots [a_{n-1}, a_n] \). We note that our
polygon is closed \(\iff a_0 = a_n\). We define the support of our polygon as simply the collection of points along our edges, or formally:

\[
|\gamma| = \bigcup_{i=1}^{n} [a_{i-1}, a_i]
\]  

(4)

**Definition (Winding Number).** Let \(\gamma\) be a directed polygon as defined above, and let \(x \notin |\gamma|\), then we can finally define the winding number, or the number of times our polygon winds around the point \(x\) as

\[
w(\gamma, x) = \frac{1}{2\pi} \sum_{i=1}^{n} \theta([a_{i-1}, a_i], x)
\]  

(5)

It may not be immediately clear that our winding number here will be an integer, but let’s prove a theorem that will assuage our doubts.

**Theorem:** Let \(\gamma\) be a closed directed polygon, and let \(x \in \mathbb{R}^2 - |\gamma|\), then our winding number will always be an integer.

**Proof:** \(\forall a_i\) we can assign a real number \(\phi_i \in \arg(a_i - x)\). Essentially we choose a real number such that our coset \([\phi_i] = \arg(a_i - x)\). Our assumption that the polygon is closed is critical as we can then require \(\phi_0 = \phi_n\). Now \(\forall i\) we note that we get a contribution:

\[
\theta([a_{i-1}, a_i], x) = \phi_i - \phi_{i-1} + 2\pi m_i; m_i \in \mathbb{Z}
\]  

(6)

So if we sum all of our terms in our winding number formula we end up with the following:

\[
2\pi w(\gamma, x) = 2\pi (m_0 + m_1 + \ldots m_n), m_i \in \mathbb{Z}
\]  

(7)

And we have confirmed that we will always have an integer for our winding number given a closed directed polygon in the plane.

### 1.4 Formal Properties of Discrete Winding Number

**Property:** Winding Number is additive.

If we let \(\gamma_1, \gamma_2\) be one cycles (a closed collection of one dimensional edges) then by definition so is \(\gamma_1 + \gamma_2\) and for some \(x \notin \mathbb{R}^2 - |\gamma_1 + \gamma_2|\) then

\[
w(\gamma_1 + \gamma_2, x) = w(\gamma_1, x) + w(\gamma_2, x)
\]  

(8)

**Property:** Winding number is locally constant

For a fixed \(\gamma\), we consider the domain \(\mathbb{R}^2 - |\gamma|\) of \(x \mapsto w(\gamma, x)\), then the function \(f(x = w(\gamma, x)\) is constant on each connected component of our domain. (we know \(f\) is continuous as it is the sum of continuous functions \(\theta\) and we also know it is integer valued, so it must be locally constant)

**Crossing Number:** if we define \(R_\phi(x) = \{p \in \mathbb{R}^2 - \{x\}; \arg(p - x) = [\phi]\} \cup \{x\}\) or \(R_\phi\) is just the ray originating at \(x\) and extending out at an angle \(\phi\), then supposing \(x \notin [a, b]\) and \(a, b \notin R_\phi(x)\) we can define the crossing number \(X([a, b], R_\phi(x))\) as simply

\[+1\text{ if } [a, b]\text{ crosses the ray in the counterclockwise direction}\]  

(9)
-1 if \([a, b]\) crosses the ray in the clockwise direction \(0\) if \([a, b]\) is disjoint from the ray

This is useful because of the Ray Escape Formula: If we have a one cycle \((\gamma = \sum_{i=1}^{n} [a_i, b_i])\) with our normal constraints, such that the ray does not meet any of the vertices of \(\gamma\) then \(w(\gamma, x) = \sum_{i=1}^{n} ([a_i, b_i], R_\phi(x))\).

Okay at this point we have built up a lot of intuition for how these winding numbers work in a 2D euclidean setting, so let’s discuss them in the context of homology.

1.5 Winding Numbers and Homology

To begin our discussion of Homology and Winding Numbers, let’s recall some basic facts about loops.

Let \(D \subset \mathbb{R}^2\), we define a loop in \(D\) as a continuous function \(f: S^1 \rightarrow \mathbb{R}^2\) such that \(f(0) = f(1)\), and we denote the set of all loops in \(D\) as simply \(\text{Loops}(D)\).

Recall from our definition of Homotopy equivalence that two loops \(f, g \in \text{Loops}(D)\) are homotopic if there exists a continuous function \(H: [0,1] \times [0,1] \rightarrow D\) with

\[
H(0, t) = f(t), \forall t \in [0,1] \quad (12)
\]

\[
H(1, t) = g(t), \forall t \in [0,1] \quad (13)
\]

\[
H(s, 0) = H(s, 1), \forall s \in [0,1] \quad (14)
\]

This leads us to some basic properties of loops and homotopy equivalence we have previously covered and is worth restating.

**Proposition**: If we let \(f, g \in \text{Loops}(D)\) and suppose that \(\forall t \in [0,1]\) the line segment \([f(t), g(t)] \subset D\), then \(f\) is homotopic to \(g\). It follows that all loops in \(\text{Loops}(\mathbb{R}^2)\) are homotopic (one can simply let \(H(s, t) = (1-s)f(t) + sg(t)\)).

However, one should be careful to note that this is not true of \(\text{Loops}(\mathbb{R}^2 - \{0\})\) for example. Consider \(f = e^{2\pi it}\) and \(g = \frac{1}{10} e^{2\pi it} - (1 + i)\).

**Definition**: We define the continuous winding number as \(\text{Loops}(\mathbb{R}^2 - \{0\}) \rightarrow \mathbb{Z}; f \mapsto w(f, 0)\) with the following properties:

\[
\text{if } f \text{ is homotopic to } g \text{ then } w(f, 0) = w(g, 0) \quad (15)
\]

\[
\text{if } f = e^{2\pi i dt} \text{ then } w(f, 0) = d \quad (16)
\]

Note that we can shift our definition to an arbitrary point \(x \in \mathbb{R}^2\) by the same method as we did in the discrete case.
Figure 1: An example of different curves and their associated winding number in the plane

**Definition**: The Mod 2 Winding Number about $\gamma$:

If we consider a simple continuous closed curve $\gamma : S^1 \rightarrow \mathbb{R}^2$, then our curve separates the plane into two connected regions. Recalling that in homology $H_0$ specifically is a measure of this connectedness, it follows that $\dim(H_0(\mathbb{R}^2 - \gamma(S^1)) = 2$. We can then think of the Mod 2 winding number of $\gamma$ about a point $x \in \mathbb{R}^2 - |\gamma|$ is simply the number which tells us whether or not $x$ is inside of $\gamma$ or not.

2 Applications of Winding Numbers

2.1 Differential Geometry

If we consider the function that parameterizes theta in the plane as simply $\theta(t) = \arctan\left(\frac{y(t)}{x(t)}\right)$, then it follows that $d\theta = \frac{1}{r^2}(xdy - ydx)$, where $r^2 = x^2 + y^2$. We can then express the winding number of a closed differentiable curve as simply:

$$w(\gamma, 0) = \oint_{\gamma} \frac{x}{r^2} dy + \frac{y}{r^2} dx$$

(17)

2.2 The Cauchy Integral Theorem

In complex analysis, the winding number is useful in applying it to Cauchy’s theorem and residue theorem. Recall that when considering $z \in \mathbb{C}$ we can equivalently define $z = x + iy$ and $z = re^{i\theta} \forall z \in \mathbb{C}$. Working with the latter form as it is much more natural with our definition of winding number, we note that $dz = e^{i\theta}dr + ire^{i\theta}d\theta$. Therefore it follows:

$$\frac{dz}{z} = \frac{dr}{r} + id\theta = d(ln(r)) + id\theta$$

(18)
As we are dealing with a closed loop, the total change in \( \ln(r) \) will be zero and we are left with the famous special case of the Cauchy integral formula:

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z} = w(\gamma, z)
\]  

(19)

We can also define the simple corollary to our theorem: if \( \gamma \) is the path defined by \( \gamma(t) = a + re^{2\pi int} \) where \( t \in [0, 1] \) then

if \( |z-a| < r \) then \( w(\gamma, z) = n \)

(20)

if \( |z-a| > r \) then \( w(\gamma, z) = 0 \)

(21)

3 Degree and Computation

3.1 A Concise Review and Definitions

Earlier, we were introduced to the notions of winding numbers (degrees) and their importance in topology and geometry. As a reminder, our winding number and, more generally, degree are defined as follows.

**Definition (Degree).** Let \( X \) and \( Y \) be closed connected oriented \( k \)-dimensional manifolds. A continuous map, \( f : X \to Y \) induces a homomorphism \( H_f \) from \( H_k(X) \) to \( H_k(Y) \). Let \([X]\) and \([Y]\) be the chosen generators (fundamental classes, i.e., isomorphisms \( \mathbb{Z} \to H_k(M;\mathbb{Z}) \)) of \( H_k(X) \) and \( H_k(Y) \), respectively; then the corresponding degree of \( f \), which is intimately related to the winding number of closed curves, is defined to be \( H_f([X]) \).

**Remark.** For winding numbers: Let \( \gamma : S^1 \to \mathbb{R}^2 - \{p\} \) for \( p \in \mathbb{R} \) be a map. Consider the induced homomorphism, \( H_\gamma : H_1(S^1) \to H_1(\mathbb{R}^2 - \{p\}) \) where \( H_k \equiv \ker(\partial_k)/\text{im}(\partial_{k+1}) \). As a result of our rank one domain and codomain, \( H_\gamma \) must correspond to multiplication by a constant, which we refer to as the degree or winding number of \( \gamma \).

3.2 Local Computations

Evidently, because the degree is a homotopy invariant, this quantity is rather crucial to geometry and applied topology. Thus, computing this quantity is often of great importance. Yet, as might be clear, these computations can be challenging; especially when considering higher-dimensional spaces or rather convoluted mappings. As a result, for us to completely understand winding numbers and degrees, we should discuss a few means—and, in particular, simple means—by which one can compute these topological numbers; namely, local computations.

**Definition (Local Degree).** Let \( f : S^k \to S^k, p \in S^n \), and \( q \in f^{-1}(p) \) be an isolated point in the inverse image of \( f \). Define the local degree of \( f \) at \( q \) to be,

\[
\deg(f, q) = \deg H_f : H_k(U, U - \{q\}) \to H_k(V, V - \{p\}),
\]

(22)

for \( U \) and \( V \) neighborhoods of \( q \) and \( p \), respectively, satisfying, \( f(U) \subset V \). As a reminder, a point \( p \in M \) is said to be an isolated point if there exits a neighborhood of \( p \) which does not contain any other points of \( M \).

Now, to see why the local degree is informative, consider the following proposition.

**Proposition.** Assume that for \( f : S^k \to S^k, p \in S^n \) has discrete inverse image, \( f^{-1}(p) = \{q_i\}_i \); then,

\[
\deg(f) = \sum_i \deg(f, q_i).
\]

(23)
At this point, many of these introduced notions may be rather intangible. Thus, consider the example topic: coloring.

Excellent.

A natural consequence of this is then, for $f$ where the isomorphisms come from excision and additivity. Yet, seeing as $n$ is the notorious Hadwiger-Nelson problem concerning a finite unit distance graph:

-how many colors are needed to color the plane so that no two points at unit distance are the same color?

Proof. As we know from our prior discussion regarding homologies from two weeks ago, the long exact sequences of the pairs $(\mathbb{S}^k, \mathbb{S}^n - Q)$ and $(\mathbb{S}^k, \mathbb{S}^k - \{p\})$ form a commutative diagram,

$$
\begin{array}{cccccc}
H_k(\mathbb{S}^k - Q) & \longrightarrow & H_k(\mathbb{S}^k) & \longrightarrow & H_k(\mathbb{S}^k, \mathbb{S}^k - Q) & \longrightarrow & H_k-1(\mathbb{S}^k - Q) \\
\downarrow H_f & & \downarrow H_f & & \downarrow H_f & & \downarrow H_f \\
H_k(\mathbb{S}^k - \{p\}) & \longrightarrow & H_k(\mathbb{S}^k) & \cong & H_k(\mathbb{S}^k, \mathbb{S}^k - \{p\}) & \longrightarrow & H_k-1(\mathbb{S}^k - \{p\})
\end{array}
$$

Naturally, the first terms in both rows and the final term in the bottom row of this diagram vanish. Furthermore, according to exactness, the lower map $H_f$ is an isomorphism. Consequently, by commutativity, we then see,

$$
\text{deg}(f) = \text{deg}(H_k(\mathbb{S}^k) \xrightarrow{H_f} H_k(\mathbb{S}^k, \mathbb{S}^k - Q) \xrightarrow{H_f} H_k(\mathbb{S}^k, \mathbb{S}^k - \{p\})).
$$

(24)

If we choose a neighborhood $V$ of $p$ so that $f^{-1}(V) = \bigcup_i U_i$ is a disjoint collection of neighborhoods of the $q_i$, then,

$$
H_k(\mathbb{S}^k, \mathbb{S}^k - Q) \cong H_k(f^{-1}(V), f^{-1}(V) - Q) \cong \bigoplus_i H_k(U_i, U_i - \{q_i\}),
$$

(25)

where the isomorphisms come from excision and additivity. Yet, seeing as $\text{deg}(f, q_i)$ is by definition the degree of the induced map $H_f : H_k(U_i, q_i) \rightarrow H_k(V, p)$, then, by additivity,

$$
\text{deg}(f) = \text{deg}(H_k(\mathbb{S}^k) \rightarrow \bigoplus_i H_k(U_i, U_i - \{q_i\}) \rightarrow H_k(V, V - \{p\})) = \sum_i \text{deg}(f, q_i).
$$

(26)

Excellent.

A natural consequence of this is then, for $f : \mathbb{S}^k \rightarrow \mathbb{S}^k$ not surjective, $\text{deg}(f) = 0$. Furthermore, we can extend this analysis to various other maps, such as a map of the form, $f : (\mathbb{D}^k, \partial \mathbb{D}^k) \rightarrow (\mathbb{D}^k, \partial \mathbb{D}^k)$. Clearly, there is an induced map, $\tilde{f} : \mathbb{S}^k \rightarrow \mathbb{S}^k$ on the quotient sphere, given by collapsing the boundary to a point, and a restriction map, $\bar{f} : \mathbb{S}^{k-1} \rightarrow \mathbb{S}^{k-1}$, given by restriction to the boundary. Moreover, using the prior reasoning, we readily find:

Lemma. For the functions constructed above, $\text{deg}(f) = \text{deg}(\tilde{f}) = \text{deg}(\bar{f})$.

Proof. Fortunately, $\text{deg}(f) = \text{deg}(\tilde{f})$ follows from naturality. As for the equivalence to $\text{deg}(\bar{f})$, consider,

$$
\begin{array}{cccccc}
0 = H_k(\mathbb{D}^k) & \longrightarrow & H_k(\mathbb{D}^k, \partial \mathbb{D}^k) & \longrightarrow & H_k-1(\partial \mathbb{D}^k) & \longrightarrow & H_k-1(\mathbb{D}^k) = 0 \\
\downarrow H_f & & \downarrow H_f & & \downarrow H_f & & \downarrow H_f \\
0 = H_k(\mathbb{D}^{k}) & \longrightarrow & H_k(\mathbb{D}^{k}, \partial \mathbb{D}^{k}) & \longrightarrow & H_k-1(\partial \mathbb{D}^{k}) & \longrightarrow & H_k-1(\mathbb{D}^{k}) = 0
\end{array}
$$

By exactness, the connecting homomorphisms $\delta$ are isomorphisms; so, by commutativity, $\text{deg}(f) = \text{deg}(\bar{f})$.

At this point, many of these introduced notions may be rather intangible. Thus, consider the example topic: coloring.

3.3 Example: Coloring

Coloring is often an interesting problem in graph theory. For example, a famous unsolved problem regarding this topic is the notorious Hadwiger-Nelson problem concerning a finite unit distance graph:

“how many colors are needed to color the plane so that no two points at unit distance are the same color?”
As a concrete example, consider the following diagram, Figure 2: A two-simplex with a subdivision; shaded triangles have three distinctly named vertices. which is a two-simplex, say \( S \), with vertices named by \{0, 1, 2\}. For \( S' \) a subdivision of \( T \), we name every new vertex of \( S' \) using any name \{0, 1, 2\} such that the following boundary condition is met: on \( \partial S' \), each vertex must not be named by the label of \( T \) on the vertex opposite that edge. As a result, we encounter the following assertion.

**Lemma (Sperner’s Lemma).** For any \{0, 1, 2\} naming of the vertices obeying the prior-mentioned boundary condition, there must exist one triangle with one of each of the three names at vertices.

**Proof.** Consider the piecewise-linear simplicial map \( f : S' \to S \), which sends each simplex in \( S' \) to the unique simplex in \( S \) determined by the convex hull of the names in \( S' \). Our boundary coloring condition implies that \( \bar{f} : \partial S' \to \partial S \) must have \( \deg(\bar{f}) = 1 \). Thus, by our previous Lemma, \( \deg(f) = \deg(\bar{f}) = 1 \). Furthermore, according to our assertion arising from connection between the degree and local degree of a map, we then realize that \( f \) must be surjective. \( \square \)

Now, even though Sperner’s Lemma is interesting and a good tool for this unique example, there is a set of ever more general theorems that extend these notions: the Brower Fixed Point, Hopf, and Borsuk-Ulam Theorems.

### 4 Borsuk-Ulam Theorems

#### 4.1 The Theorems

In the previous sections, we examined some of the interesting consequences corresponding to the degree of functions: specifically, those whose domain is the \( n \)-disk, or its boundary, the \( n \)-sphere. At this point, we should thus assert some groundbreaking theorems, mentioned earlier, which push these conceptualizations even further.

**Theorem (Borsuk-Ulam Thoerem).**

1. For every continuous function \( f : \mathbb{S}^n \to \mathbb{R}^n \), there exists a point \( x \in \mathbb{S}^n \) with \( f(x) = f(-x) \).
2. For every antipodal function \( f : \mathbb{S}^n \to \mathbb{R}^n \), there exists a point \( x \in \mathbb{S}^n \) with \( f(x) = 0 \).
3. No antipodal mapping \( f : \mathbb{S}^n \to \mathbb{S}^{n-1} \) exists.
4. No mapping \( f : \mathbb{B}^n \to \mathbb{S}^{n-1} \) that is antipodal on the boundary \( \partial \mathbb{B}^n = \mathbb{S}^n \) exists.

**Remark.** Any points \( x_1, x_2 \in \mathbb{S}^n \) are referred to as antipodal if \( +x_1 = -x_2 \). Likewise, a function \( f : \mathbb{S}^n \to M \) is antipodal if \( f \) is continuous and, for every \( x \in \mathbb{S}^n \), \( f(-x) = -f(x) \).
Proposition. Every one of the above assertions is equivalent.

Proof. 
(1a) ⇒ (1b): For any antipodal function \( f : \mathbb{S}^n \rightarrow \mathbb{R}^n \), there exists a point \( x \in \mathbb{S}^n \), which satisfies the relationship, 
\( f(x) = f(-x) = -f(x) \). Consequently, \( f(x) = 0 \).
(1b) ⇒ (1a): By (1b), consider the antipodal function \( g : \mathbb{S}^n \rightarrow \mathbb{R}^n \), \( x \mapsto f(x) - f(-x) \) for antipodal function \( f \).
(1b) ⇒ (2a): An antipodal function \( f : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1} \) gives an antipodal function from \( \mathbb{S}^n \) to \( \mathbb{R}^n \), which is nowhere zero, since \( 0 \not\in \mathbb{S}^{n-1} \) and \( \mathbb{S}^{n-1} \subset \mathbb{R}^n \). Yet, such a function is prohibited by (1b).
(2a) ⇒ (1b): A nonzero antipodal function \( f : \mathbb{S}^n \rightarrow \mathbb{R}^n \) gives \( g : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1} \), \( x \mapsto f(x) / \|f(x)\| \) contradicting (2a).
(2b) ⇒ (2a): \( \pi : (\mathbb{S}(+))^n \rightarrow B^n, (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n) \) is a homeomorphism. Thus, an antipodal map \( f : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1} \) would give a function \( g : B^n \rightarrow \mathbb{S}^{n-1} \) that is antipodal on the boundary of \( \partial B^n \) according to \( g(x) \equiv f(\pi^{-1}(x)) \), which naturally contradicts (2b).
(2a) ⇒ (2b): A map \( g : B^n \rightarrow \mathbb{S}^{n-1} \) that is antipodal on the boundary \( \partial B^n \) gives a function \( f : \mathbb{S}^n \rightarrow \mathbb{S}^{n-1} \) by \( f(x) \equiv g(\pi(x)) \) and \( f(-x) \equiv -g(\pi(x)) \) for \( x \in (\mathbb{S}(+))^n \). However, this is antipodal, which contradicts (2a). \( \square \)

Wonderful. At this point, this theorem should start becoming familiar. Yet, to make one final connection, recall the Brouwer Fixed Point Theorem discussed previously in this course.

Theorem (Brouwer Fixed Point Theorem). For every continuous function \( f : B^n \rightarrow B^n \), there exists a point \( x \in \mathbb{S}^n \), such that \( f(x) = x \), i.e., \( x \) is a fixed point.

Proof. Assume there exists a continuous function \( f : B^n \rightarrow B^n \) such that \( f \) has no fixed points. Define \( g : B^n \rightarrow \mathbb{S}^{n-1} \) such that \( g(x) \) is the point on \( \mathbb{S}^{n-1} \) that intersects with the ray from \( f(x) \) to \( x \). Naturally, this is a well-defined retraction since there is no fixed point at which the function would not be well-defined. But, since the identity is clearly antipodal, i.e., \( g(-x) = -g(x) \) for \( x \in \mathbb{S}^{n-1} \), such a function contradicts (2b). Easy! \( \square \)

4.2 A Theor-YUM

In conclusion, we now wish to address a perhaps more important, and certainly more comical, consequence of the Borsuk-Ulam Theorems: the Ham Sandwich Theorem. Yet, we must first quickly understand a few conceptualizations.

Definition (Hyperplane). A hyperplane is the set, \( \{x \in \mathbb{R}^n : \langle a, x \rangle = b \} \) for some \( a \in \mathbb{R}^d \) and \( b \in \mathbb{R} \).

Definition (Finite Borel Measure). A finite Borel measure \( \mu \) on \( \mathbb{R}^d \) is a measure on \( \mathbb{R}^d \) such that every open subset of \( \mathbb{R}^d \) are measurable and \( 0 < \mu(\mathbb{R}^d) < \infty \).

Theorem (Ham Sandwich Theorem, or Stone-Tukey). Let \( \mu_1, \ldots, \mu_d \) be finite Borel measures on \( \mathbb{R}^d \) such that every hyperplane has measure 0 for each of the \( \mu_i \); then there exists a hyperplane \( h \) such that,

\[
\mu_i(h^+) = \frac{1}{2} \mu_i(\mathbb{R}^d) \quad \text{for} \quad i = 1, \ldots, d, \tag{27}
\]

where \( h^+ \) is one of the half-spaces defined by \( h \).
Figure 3: A sandwich portraying the Ham Sandwich Theor-YUM.

5 Conclusion

References
