Manifolds, Configuration Space, and $\mathbb{C}P^1$

Kohtaro Yamakawa and Saveliy Yusufov

February 7, 2019

Contents

1 Manifolds 1
   1.1 Definition . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
   1.2 Atlas and charts . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2

2 Configuration space of linkages 2
   2.1 In Robotics . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
      2.1.1 Robot manipulators . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
   2.2 In Physics . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
      2.2.1 As a phase space . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
      2.2.2 As a Hilbert space . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3

3 Complex Projective Line and the Riemann Sphere 4
   3.1 Definition . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
   3.2 Mobius Transformations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4

4 Bloch Space as a "configuration space" 4

1 Manifolds

1.1 Definition

A manifold can be defined as a topological space that is:

Definition 1.1. n-dimensional topological Manifold

A topological space $X$ is an $n$ dimensional manifold if it is:

- second countable Hausdorff space
- locally homeomorphic to $\mathbb{R}^n$

Let’s break this definition down.

Definition 1.2. Second Countable

A topological space is second countable if it has a countable basis.

Definition 1.3. Hausdorff Space

A topological space $X$ is called Hausdorff if $\forall x, y \in X$ there exist open neighborhoods of each that are disjoint of each other. In other words, $\exists U, V \in T$ such that $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$.

We have already review the definition of a homeomorphism in class. It is the notion of a continuous function, which deforms topological spaces into other ones without 'breaking' or 'ripping' the space. We begin by first explaining what this term locally homeomorphic means rigorously.

Definition 1.4. Locally Homeomorphic

A topological space $X$ is locally homeomorphic if, for any point $x$ in our topological space $X$, there exists an open neighborhood $U$ of $x$ such that $U \simeq \mathbb{R}^n$.

Put in more conceptual terms, up to some stretching and squeezing, a manifold is something that if you zoom in really close, it looks like $R^n$, like space that we are used to. In essence, it means that what we are looking at isn’t really weird. Just for completion, we define a topology on a manifold as,

A subset $W$ of the manifold $M$ is open if for every $x \in W$ there is a chart with domain $U$ such that $x \in U \subset W$.

Let’s begin with a few examples of manifolds

- $\mathbb{R}^n$
- $\mathbb{S}^n$
- Mobius Strip ( take a look at its quotient space!)

and things that are not a manifold
• Cantor set: deleting the middle third and intersecting all of the these points.

• Double Cone

• Line with two origins. \[ \mathbb{R} \times \{a\} \cup \mathbb{R} \times \{b\} / ((x,a) \sim (x,b), x \neq 0) \]

(why are these not manifolds?)

1.2 Atlas and charts

Now, in particular, these manifolds are nice because we can determine a coordinate on them. In other words, we can do calculus and all the nice things we do in \( \mathbb{R}^n \) on these manifolds.

Definition 1.5. Coordinate chart

For a topological \( n \)-manifold \( M \), a (coordinate) chart is a pair \((U, \phi)\) with a map \( \phi : U \to V \) where \( U \) is an open set of \( M \) and \( V \) is an open set in \( \mathbb{R}^n \) which must be a homeomorphism.

Remarks

• A collection of charts which covers a manifold is an atlas. They are not unique.

• A manifold is called differentiable if for any charts \((U_\alpha, \phi_\alpha)\) and \((U_\beta, \phi_\beta)\), the composition (or transition map \( \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \) is differentiable. (as it goes from \( \mathbb{R}^n \) to \( \mathbb{R}^n \)).

• A manifold is called smooth if the transition map is smooth.

• A manifold is complex if the charts send to \( \mathbb{C}^n \) and the transition maps are holomorphic. (complex differentiable)

So the great thing about using manifolds is that we are able to talk about a point on a manifold with local coordinates defined by the charts. Essentially these charts give us an idea of how to look at the manifold in some structured way and perhaps calculate or define things on them through \( \mathbb{R}^n \) that wouldn’t make sense otherwise.

We delve into some more complex examples to see how they are used and even some of their physical applications.

2 Configuration space of linkages

2.1 In Robotics

In robotics, a configuration describes the exact state of the robot, in space, and the corresponding configuration space, \( C \) is the set of all possible configurations (states) of the robot. A basic motion planning problem in robotics is to compute a continuous motion that connects a start configuration \( C_{\text{start}} \) to an end configuration, \( C_{\text{end}} \). As the motion is carried out, the motion must also be planned such that it avoids obstacles. Given a robot and an obstacle, the geometric representation of these two objects can either be in \( \mathbb{R}^2 \) or in \( \mathbb{R}^3 \). However, the continuous motion will be represented as a path in a configuration space of up to \( n \) dimensions.

1. The simplest case would be a robot that can be represented as a single point in \( \mathbb{R}^2 \) – this is the robot in real-life or in its “workspace”. If the robot would only be allowed to translate through space, we would a configuration space, \( C \), that is a 2D plane. The robot’s configurations can be represented using two parameters, i.e., as an ordered pair \((x, y)\).

2. If the robot is a 2D shape that can translate and rotate, the workspace is still 2-dimensional. However, \( C \) is the special Euclidean group \( SE(2) = \mathbb{R}^2 \times SO(2) \) (where \( SO(2) \) is the special orthogonal group of 2D rotations), and a configuration, \( c \) can be represented using 3 parameters, as such \( c = (x, y, \theta) \).

3. If the robot is a solid 3D shape that can translate and rotate, the workspace is 3-dimensional, but \( C \) is the special Euclidean group \( SE(3) = \mathbb{R}^3 \times SO(3) \), and a configuration requires 6 parameters: \((x, y, z)\) (for translation), and the corresponding angles, \((\alpha, \beta, \gamma)\).

4. If the robot is fixed at a point in its workspace with \( n \) arm linkages, then its configuration space, \( C \), will consist of \( n \) dimensions. For a robot/configuration space demo, feel free to clone and run an interactive visualization tool we made! Available on GitHub: https://github.com/smu160/Math_Seminar.
### 2.1.1 Robot manipulators

We can compute the position of a robot end effector as follows:

First, we set the following variables:

\[
\begin{align*}
c_\alpha &= \cos(\alpha), \quad s_\alpha = \cos(\alpha) \\
c_\theta &= \cos(\theta), \quad s_\theta = \cos(\theta) \\
c_{\alpha+\theta} &= \cos(\alpha + \theta), \quad s_{\alpha+\theta} = \cos(\alpha + \theta),
\end{align*}
\]

\(\alpha\) is the angle between the robot base and the first arm linkage, and \(\theta\) is the angle between the first arm linkage and the second arm linkage.

Now, we let \(L_1\) be the length of the first arm linkage and we let \(L_2\) be the length of the second arm linkage. Finally, using all the variables we defined, we can compute the configuration of the robot in its workspace as follows:

\[
\begin{bmatrix}
L_1c_\alpha \\
L_1s_\alpha \\
L_2c_{(\alpha+\theta)} \\
L_2s_{(\alpha+\theta)}
\end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

### 2.2 In Physics

In physics, the configuration space as a slightly different meaning. It is the set of points that a system can possibly hold. So, if ball moved with position \(x\) then the configuration space would plot this \(x\) as a function of some parameter. Or, we could graph this ball’s position with on the order of \(10^{23}\) atoms its made, \(\{x_n\}\) and consider a \(10^{23}\) dimensional configuration space. Pick your poison!

#### 2.2.1 As a phase space

However, in physics, we don’t just care about the position, we care about the velocity! The configuration space is simply not enough to properly describe a mechanical system. Is there a natural way to extend this configuration space? In fact, there is and it is defined in a very rigorous mathematical way. Let’s first give some introduction in the notion of tangent bundles and cotangent bundles.

**Definition 2.1. Tangent spaces**

For an \(n\) dimensional (compact \(C^k, k \geq 1\)) manifold \(M\), the tangent plane is the set of tangent vectors at a particular point \(x \in M\). More rigorously, it is the equivalence class of curves passing through \(x\) modulo the derivatives at the point \(x\) coincide.

**Definition 2.2. Tangent Bundle**

For a compact manifold \(M\), the tangent bundle is the disjoint union of the tangent spaces at each point \(x \in M\),

\[TM = \bigcup_{x \in M} T_x M\]

So, in considering the manifold of the configuration space, a generic point on this tangent bundle is a point \((q, \dot{q})\), which describes a configuration of the physical system at this location \(q\) with velocity \(\dot{q}\).

But for those more physics inclined, we MOSTLY care about a similar phase space that is the momentum phase space which has points \((q, p)\). This too is present in the mathematical formulation as the DUAL of the tangent bundle, the cotangent bundle.

**Definition 2.3. Cotangent space**

The cotangent space is a vector space attached to every point \(x\) of a smooth manifold defined as the dual space of the tangent space. In other words, we assign to every point a function which takes points \((q, \dot{q}) \in T_p M\) and send them to a real number.

\[\alpha \in T^*_x M, \text{ such that } \alpha : T_x M \to \mathbb{R}\]

**Definition 2.4. Cotangent bundle**

For a smooth manifold \(M\), the cotangent bundle \(T^* M\) is the disjoint union of the cotangent spaces \(T^*_x M\).

The set of positions and momenta are then the cotangent bundle \(T^* Q\) of the configuration manifold \(Q\). Note that in this perspective, the momentum vectors are linear functionals of the tangent plane, i.e. they take vectors in the tangent plane and give back a real number. That comes down to the fact that we define a momentum as \(p = \sum p_i dq^i\) where \(dq^i\) are covectors themselves. I don’t want to get into too many specifics but essentially overall what we have shown is that the vector phase space and the momentum phase space are themselves manifolds! So, the the space we describe just positions, the space we describe position and velocities, and the space we describe position and momenta are manifolds:

This allows us to do all kinds of cool things like consider a natural function \(H\) of energy on this phase space without worrying about odd things happening. (or think of things as symplectic!!! whattt)

#### 2.2.2 As a Hilbert space

In quantum mechanics, it turns out that we need way more degrees of freedom to describe a particle. Our configuration space is analogized and becomes the state space, where we can think of point particles living on \(CP^1\) and \(n\) particles on \(CP^n\). More to follow on what this manifold actually is.
3 Complex Projective Line and the Riemann Sphere

3.1 Definition

**Definition 3.1. Riemann Sphere** The Riemann Sphere is the extended complex numbers, i.e. the one point compactification of the complex numbers. (It is the complex plane and we add an point so that topology still makes sense).

To show it is a one dimensional complex manifold, we consider two charts from the complex number plane \( \mathbb{C} \) to the riemann sphere. For a complex number \( \zeta \) in a complex plane \( \mathbb{C} \) and \( \zeta \) in another complex plane \( \mathbb{C} \), we identity \( \zeta \sim \frac{1}{\zeta} \). Then the maps \( f(z) = \frac{1}{z} \) is the transition map between these two charts, gluing them together.

The transition maps tell us how to form the Riemann sphere from two complex planes. We have glued them together such that every point in one complex plane is directly related to the other except the origin. So the origin in one of the planes is associated with infinity in the other and visa versa.

It turns out that the riemann surface can alternatively be described as,

**Definition 3.2. Complex Projective Line**

The complex projective line can be defined as the following,

\[
\mathbb{C}P^1 = \mathbb{C}^2 / (\alpha, \beta) \sim (\lambda \alpha, \lambda \beta)
\]

In other words, it is the projective space of all complex lines. We can see how the complex projective line is in fact the riemann sphere by noting that if it doesn’t matter what \( \lambda \) is, then we can go ahead and set one of the coordinates to 1, Then,

\[
(\alpha, \beta) = (\xi, 1) = (1, \zeta)
\]

But then,

\[
(\xi, 1) = (1, \frac{1}{\zeta}) = (1, \zeta)
\]

which means that we have the same identification that every point is determined by the transition map \( \zeta = \frac{1}{\xi} \). We have just shown that the complex projective line is just another way of stating the Riemann surface, because both are maps from two complex planes in such a way that \( \zeta = \frac{1}{\xi} \).

3.2 Mobius Transformations

The group of automorphisms, are the set of isomorphisms which have the same domain and range. They preserve the structure of the manifold in question. In the context of complex manifolds, the group of automorphisms are the group of biholomorphisms (or bijective holomorphic functions, bijective complex differentiable functions) from the complex manifold to itself. It turns out that any such map can be written as a mobius transformation.

**Definition 3.3. Mobius Transformation**

The mobius transformations are defined on the riemann sphere as,

\[
f(\zeta) = \frac{a\zeta + b}{c\zeta + d}
\]

such that \( ad - bc \neq 0 \).

They can alternatively be defined on the complex projective line as,

\[
f(\alpha, \beta) = (a\alpha + b\beta, c\alpha + d\beta) = (\alpha \quad \beta) \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]

So, the mobius transformations can be described as \( 2 \times 2 \) complex matrix with nonzero determinant.

We remark that in the case of the later transformation, two matrices are the same mobius transformation iff they differ by a nonzero factor. This is quite easy to see!

We then find that the "projectiveness" of the complex projective line makes the mobius transformations themselves have this property. In fact, the mobius transformations thus make a group called the projective linear transformations \( PGL(2, \mathbb{C}) \) (which just means that they are \( 2 \times 2 \) matrices made of complex numbers in their slots that are the same up to a scale \( \lambda \).

Now this is all nice and all, but some people may be bored. Where is the physical applicability?

4 Bloch Space as a "configuration space"

Let's consider a quantum mechanical system with two-levels. either we are at the ground state 0 or at the not-ground state 1. Then, any wavefunction can be represented as,

\[
\psi = a0 + b1
\]

for complex numbers \( a, b \in \mathbb{C} \). It turns out that this two level system can be represented by complex projective line. Going into the details of why this is so kind of requires me to get into the details about quantum mechanics so I leave that to you. However! any sort of operation on the wavefunction should give us back a wavefunction right? And in that sense, any operation in these qubits should be representable by some operation on the riemann sphere right? Well it turns out, according to this paper "Qubit geometry and conformal mapping" by Jae-ween Lee and Chang Ho Kim, any (unitary) operation on qubits are precisely the conformal mobius transformations (up to a phase factor). So, it turns out that quantum mechanics and manifolds go quite along!!)

4