Introduction to Homology

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1 Homology

What is Homology? To put it simply, we use Homology to count the number of n dimensional holes in a topological space! In general, our approach will be to add a structure on a space or object (and thus a topology) and figure out what subsets of the space are cycles, then sort through those subsets that are holes. Of course, as many properties we care about in topology, this property is invariant under homotopy equivalence. This is the slightly weaker than homeomorphism which we before said gave us the same fundamental group.

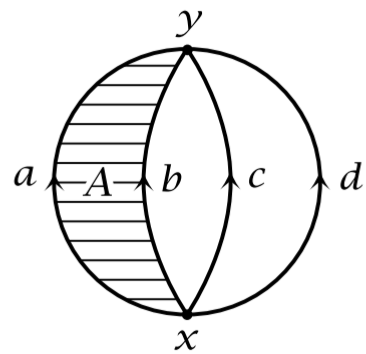

Figure 1: Hatcher p.100

Just for reference to you, I will simply define the nth Homology of a topological space X.

$$H_n(X) = \ker \partial_n / \mathrm{Im}\partial_{n-1}$$

which, as we have said before, is the group of n-holes.

1.1 Simplices: a Review

Just for your sake, we review what standard K simplices are, as embedded inside (or living in) $\mathbb{R}^{k+1}$

$$\Delta^k = [v_0, \dots, v_k] = \left\{ \sum_{i=0}^{n} x_i v_i \middle| \text{ such that } \sum x_k = 1 \right\}$$

For example, the 0 simplex is a point, the 1 simplex is a line, the 2 simplex is a triangle, the 3 simplex is a tetrahedron. Remember, that the condition on the right hand side, tells us that no matter how big k gets, we are always talking about the surface, a k dimensional objective in a $k+1$ dimensional space. So, when I say a 3 simplex is a tetrahedron, then I mean that we are not only considering the shell of this tetrahedron but also the inside, because that is the 3 dimensional object inside a 4 dimensional space R^4, where it may help to think of the 4th dimension as time. Don't ask me what the higher dimensional simplicies look like because they ain't *simple*[1].

From here, we can consider a n-simplex which are like standard simplices but the vectors above are not unit vectors of a $\mathbb{R}^{n+1}$ space but are $n+1$ affinely independent points, which our bois Abhineet and Noah showed for us and I won't go over. (essentially, for a 2 simplex, we can't pick 3 colinear points so that it looks like a line)

Now, we know that we can construct many objects from these simplices, which we will call complexes, which just comes from gluing together any set of simplices together and letting the unit vectors v_0 be arbitrary vectors in an arbitrary space R^n.

1.2 Δ Simplices: not a Review

To define the Δ simplex, we must first define the *Canonical linear homomorphism.* Recall that when we were talking about a general n simplex and a standard n simplex, we considered the former as a mapping of the basis vectors of the standard n simplex to affinely independent vectors. That transformation is this homomorphism! Put more explicitly, the Canonical linear homomorphism is one that maps a standard n simplex Δ^n, to the n simplex,

$$\sigma^n : \Delta^n \to [v_0, \dots, v_n]$$

[1] pun

So, for a (vector) point in Δ^n, $(x_0, x_1, \dots, x_n)$, from the above definition,

$$\vec{x} = (x_0, x_1, \dots, x_n) \to \sum_{i=0}^{n} x_i v_i$$

Now, a Δ complex will be similarly constructed as a sequence of Δ simplices with the following stipulation. A Δ-complex structure on a topological sapce X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ for each n such that,

1. $\sigma_\alpha\big|_{\Delta^{n\circ}}$ is injective and each point of X is in the image of exactly one $\sigma_\alpha\big|_{\Delta^{n\circ}}$.

2. For each i,

$$\begin{tikzcd}
[v_0, \dots, \hat{v}_i, \dots, v_n] \arrow[d, "\sigma_\alpha"] & \Delta^{n-1} \arrow[l] \arrow[dl, "\sigma_\beta"] \\
X &
\end{tikzcd}$$

   where σ_β is one of the maps ni the Δ complex structure. This just lets us know that the face sand edges of faces are all consistent.

3. $A \subset X \iff \sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ^α. This states that X is the quotient space,

$$X = \frac{\coprod_{n,\alpha} \Delta_\alpha^n}{d_i : \Delta_\alpha^n \sim \Delta_\beta^{n-1}}$$

   The quotient identifies the points near the edge of the faces to the edges.

In conjunction with this, we define the set of n simplices,

$$S_n(X) = \{\sigma_\alpha : \Delta^n \to X\}$$

and of the chain group,

$$C_n^\Delta(X) = \Delta_n(X) = \left\{ \sum \sigma_\alpha \in S_n(X) a_\alpha \sigma_\alpha \middle| a_\alpha \in \mathbb{Z} \right\}$$

where the latter is the free abelian group generated by the elements of $S_n(X)$

1.3 Boundary Operator

Now, we identify $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is a linear homomorphism,

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha\big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

Let's do an example to calculate this:

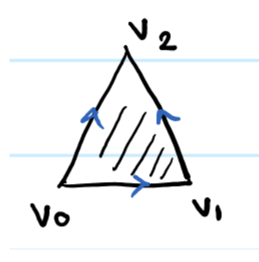

Figure 2: Alishahi Notes

$$\partial(\sigma) = \sigma\big|_{[v_1,v_2]} - \sigma\big|_{[v_0,v_2]} + \sigma\big|_{[v_0,v_1]}$$

In addition, lets show $\partial_{n-1} \circ \partial_n = 0$.

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha\big|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

Then,

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \sum_{i=0}^{n} (-1)^i \partial_{n-1}\sigma\big|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} \\
&= \sum_{i=0}^{n} (-1)^i \left(\sum_{j<i} (-1)^j \sigma\big|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} + \sum_{j>i} (-1)^{j-1} \sigma\big|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} \right) \\
&= \sum_{j<i} (-1)^{i+j} \sigma\big|_{[v_0,\ldots,\hat{v}_j,\ldots,\hat{v}_i,\ldots,v_n]} + \sum_{j>i} (-1)^{i+j-1} \sigma\big|_{[v_0,\ldots,\hat{v}_i,\ldots,\hat{v}_j,\ldots,v_n]} \\
&= 0
\end{aligned}$$

Therefore, Im $\sigma_{n+1} \subset \ker \partial_n$. We call all elements of the $\ker \partial_n$ as n cycles and Now, we can define homology.

1.4 Simplicial Homology: DEF not a Review

For a chain complex $(C_*, \partial_*) = \{C_n, \partial_n\}$ such that,

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Then, $H_n = \frac{\ker \partial_n}{\text{Im } \partial_{n+1}}$. Let's now revisit our original example and figure out the Delta complexes.

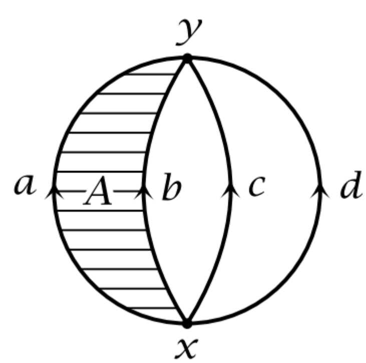

Figure 3: Hatcher p.100

What we SHOULD get is

$$H_0(X) = 0, \qquad\qquad H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$$

where the direct sum, is just the direct product except the addition operation is used. We notice that the homology group doesn't exactly count the number of holes, but every hole has a copy of $\mathbb{Z}$.
What we have just shown is how to define homology on a Δ complex, a small subset of all the possible spaces. However, we can extend this to all topological spaces.

1.5 Singular Homology

This is a natural extension of simplicial homology which extends the idea beyond Δ complexes to general topological spaces. However, it is relatively harder to calculate homology groups in this manner, (and the groups are equivalent) and we don't want our hands to get too dirty (or bore) you so we will move on.

2 Higher Homotopy Groups and Hurweicz Theorem

Now, if we recall the fundamental group, and consider that of S^1, we get a fundamental group to be $\mathbb{Z}$. Now that is quite interesting because the reason why the fundamental group is $\mathbb{Z}$ is because of the *hole* in the middle. So, it turns out there is a connection between 1 holes and the first homology group! Here, we will first define higher homotopy groups.

$$\pi_n(X) = \langle S^n, X \rangle = \pi_0 \left(\mathrm{Hom}_{\mathrm{Top}_*}(X, Y) \right)$$

where $\pi_0(X)$ is the set of path components of X. We can deconstruct this definition by considering the examples of $n = 0, 1, 2$.
Now, we say a space is n connected, if the first n homotopy groups are trivial (which means that a space is 0 connected iff it is non empty and path connected) The Hurweicz theorem states that if X is $(n-1)$ connected the hurweicz map, defined as, $h)* : \pi_k(X) \to H_k(X)$ (which for $k = 1$ is the abelianization projection map, $h_* : \pi_1(X) \to \pi_1(X)/[\pi_1(X), \pi_1(X)]$) is an isomorphism for $k \leq n$ when $n \geq 2$. In the case While this connection is hard to really care about, for the case of $k = 1$, we find that this implies that the first homology group of a path connected space is the abelianization of the fundamental group.

$$H_1(X) \cong \mathrm{Ab}\pi_1(X)$$

Let's check this for $S^1 \vee S^1$.
So why do we care? Turns out for a map between CW Complexes, if the map induces an isomorphism between the homotopy groups, then groups are homotopy equivalent. Thus, holes DO completely identify spaces (for CW Complexes) and are good property to care about.

3 Exact Sequences

3.1 Key Definitions

For this section we will use the notation $\mathcal{A}$ to refer to some collection of algebraic objects. Specifically, one should think of $\mathcal{A}$ as either the category of groups, rings, or modules. If you're familiar with the notation, we can more generally let $\mathcal{A}$ be any abelian category.

Definition. Let A be the sequence

$$\cdots \to A_n \to A_{n-1} \to \cdots \to A_1 \to \cdots$$

where the $A_i \in \mathcal{A}$ for all i and for all i we have a map $\phi_i : A_i \to A_{i-1}$. Then A is called an *exact sequence* if $\ker \phi_i = \mathrm{im}\phi_{i+1}$ for all i.

Remark 3.1.1. Note every exact sequence is a chain complex. In fact, the chain complex condition $\phi_i \circ \phi_{i+1} = 0$ can be reformulated as $\mathrm{im}\phi_{i+1} \subset \ker \phi_i$.

Definition. An exact sequence S is called a *short exact sequence* if it is of the form:

$$0 \to A \to B \to C \to 0$$

for some $A, B, C \in \mathcal{A}$.

Example 3.1.1. The infinite sequence:

$$\cdots \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \cdots$$

is exact where every element is $\mathbb{Z}/4\mathbb{Z}$ and every map is multiplication by 2. Additionally the following sequence:

$$0 \to p\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

is short exact.

Example 3.1.2. If $A, B \in \mathcal{A}$ and $\phi : A \to B$ is a morphism then

$$0 \to \ker \phi \to A \to \mathrm{im}\phi \to 0$$

is a short exact sequence, where the map $\ker \phi \to A$ is inclusion and the map $A \to \mathrm{im}\phi$ is ϕ.

Remark 3.1.2. The first isomorphism group theorem equivalently states that if $\phi : G \to H$ is a group map then

$$0 \to \ker \phi \to G \to G/\ker \phi \to 0$$

is a short exact sequence.

3.2 Recreating Groups From Exact Sequences

As a forewarning, what we'll be doing in this section of the talk will now be rigorous, but we're including it since it is a helpful piece of intuition for what follows. The idea that we will discuss in this section is that given an exact sequence

$$0 \to A \to B \to C \to 0$$

for $A, B, C \in \mathcal{A}$, if we understand the structure of the maps $A \to B$ and $B \to C$, and fully understand 2 of the 3 out of A, B, C then we essentially can fully understand the 3rd. We're purposefully being vague about the definition of "$understand''$ as it is hard to define in general, but the idea is that when looking at homology groups $H_n(A), H_n(B)$ we can often times get an induced map $H_n(A) \to H_n(B)$ from the map $A \to B$. However, H_n is a quotient of two groups so we might know what representatives for the cosets are but it might be hard to determine when two elements are equal or what the product of two elements are. So the significance of this idea is that we can recreate our homology group by putting it in an exact sequence of the form:

$$0 \to H_n(A) \to H_n(B) \to H_n(C) \to 0$$

Since we don't have a rigorous statement of what we mean by understand and to avoid getting into the details of category theory, we'll detail this with some examples using groups. Suppose we have an exact sequence of the form above. Then we can make the following three conclusions:

1. Suppose we know B, C. Then A is isomorphic to the kernel of the map $B \to C$

2. Suppose we know A, B. Then C is isomorphic to B/A (where we view A as the image of the injective $A \to B$ map).

3. Suppose we know A, C and we have some way of viewing C as a subgroup of B. To be more specific let $\phi : B \to C$ and suppose we have a map $\alpha : C \to B$ such that $\alpha \circ \phi$ is the identity map. Then note that $\alpha(C)A = B$ as $C \cong B/A$. Furthermore, A is isomorphic to $\ker \phi$ and thus is a normal subgroup of B. By definition then $B \cong A \rtimes C$, that is the semidirect product of A and C.

In fact we can extend this idea above. If we have

$$A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} D \xrightarrow{d} E$$

and we understand the structure of all the maps and know what A, B, D, E are as algebraic objects then we can construct from the above the following 3 exact sequences:

$$\begin{aligned}
0 \to \ker b \to B \to \ker c \to 0 \\
0 \to \ker c \to C \to \mathrm{im} c \to 0 \\
0 \to \mathrm{im} c \to D \to \mathrm{im} d \to 0
\end{aligned}$$

As we understand B, D and the maps we can use the first and third exact sequences to understand $\ker c, \mathrm{im} c$. From here, we can use the second short exact sequence to recreate C.

4 Long Exact Homology Sequences

The goal of this section will be to show how we can compute a large homology group $H_n(X)$ from smaller homology groups. The key idea will be to find a long exact sequence containing $H_n(X)$ where we understand the maps and 2/3 of the objects. Using the ideas of the previous section we can essentially recover $H_n(X)$.

4.1 Exact Sequences of Chain Complexes

Definition. Let A, B, C be three chain complexes such that there exists maps $a_n : A_n \to B_n$ and $b_n : B_n \to C_n$. We say that

$$0 \to A \to B \to C \to 0$$

is an exact sequence if for all n

$$0 \to A_n \to B_n \to C_n \to 0$$

is exact and the following diagram commutes:

$$\begin{tikzcd}
A_n \arrow[r] \arrow[d, "a_n"] & A_{n-1} \arrow[d, "a_{n-1}"] \\
B_n \arrow[r] \arrow[d, "b_n"] & B_{n-1} \arrow[d, "b_{n-1}"] \\
C_n \arrow[r] & C_{n-1}
\end{tikzcd}$$

The key theorem that allows us to create the long exact sequences we want is the following:

Theorem 4.1. *Suppose*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of chain complexes. Then the following sequence is exact:

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{a_n^*} H_n(B) \xrightarrow{b_n^*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{a_{n-1}} \cdots$$

where a_n^*, b_n^* are the maps induced by a_n, b_n and ∂ is some constructible map.

Remark 4.1.1. For those of you more familiar with category theory, ∂ can be constructed using the snake lemma.

Example 4.1.1. For instance if we have the following exact sequence of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "id"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d, "\times 2"] & 0 \arrow[d] \\
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "id"] \arrow[d] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

It gives rise to the exact sequence:

$$0 \to 0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

4.2 Relative Homology Groups

Definition. Let X be a space and A a subspace of X. Let $C_n(X), C_n(A)$ be the free abelian groups of singular n-chains and set:

$$C_n(X, A) = C_n(X)/C_n(A)$$

Then the *relative homology* group $H_n(X, A)$ is the homology of the chain complex

$$\cdots \to C_{n+1}(X, A) \to C_n(X, A) \to C_{n-1}(X, A) \to \cdots$$

Theorem 4.2. *If X is a space and A is a subspace of X, then there is the long exact sequence of homology:*

$$\cdots \xrightarrow{\partial} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots \longrightarrow H_0(X, A) \longrightarrow 0$$

Proof. As $C_n(X, A) \cong C_n(X)/C_n(A)$ we can construct the short exact sequence of chain complexes:

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

Applying the theorem from previous section gives the desired result exactly. □

This gives us an exact sequence as we wanted! However, we still need to be able to calculate the expression $H_n(X, A)$. This is where the excision theorem comes in:

4.3 The Excision Theorems

Theorem 4.3. *Let X be a space with subspaces U, A such that the closure of U is in the interior of A. Then:*

$$H_n(X, A) \cong H_n(X - U, A - U)$$

Example 4.3.1.

We can actually use the excision theorem to relate $H_n(X, A)$ to $H_n(X/A)$ in certain instances.

Definition. Suppose $C_n(X)$ is an R-module for all n. Then the *reduced homology* $\tilde{H}_n(X)$ is the homology of the complex:

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots C_1(X) \to C_0(X) \to R \to 0$$

Remark 4.3.1. In particular for all $n > 0$, $H_n(X) \cong \tilde{H}_n(X)$.

Remark 4.3.2. As $R/R \cong 0$, in 4.2 we can actually use the same method to get an identical exact sequence just of the reduced homology groups with $\tilde{H}_n(X, A) = H_n(X, A)$.

Lemma 1. *Let X be a space and $p \in X$ a basepoint. Then for all $n \geq 0$,*

$$H_n(X, \{p\}) \cong \tilde{H}_n(X)$$

Proof. We can see from definitions that $\tilde{H}_n(\{p\}) = 0$ for all n. Now consider the long exact sequence of relative homology obtained by setting $A = \{p\}$. For $n > 0$ we have:

$$0 = \tilde{H}_n(\{p\}) \to \tilde{H}_n(X) \to \tilde{H}_n(X, \{p\}) \to \tilde{H}_{n-1}(\{p\}) = 0$$

is exact. Thus

$$H_n(X, \{p\}) \cong \tilde{H}_n(X, \{p\}) \cong \tilde{H}_n(X)$$

For $n = 0$ similar reasoning applies. □

Theorem 4.4. *Suppose A is a closed subset of a cell complex X. Then for all $n > 0$,*

$$H_n(X, A) \cong \tilde{H}_n(X/A)$$

Proof. (Sketch) First show there is an isomorphism:

$$H_n(X, A) \cong H_n(X/A, A/A)$$

and then apply the preceding lemma □

Remark 4.3.3. This allows us to get an exact sequence using $H_n(X/A)$ instead of $H_n(X, A)$ for certain pairs (X, A)!

4.4 Mayer-Vietoris Sequence

One last exact sequence that can be useful in computing homology groups is the Mayer-Vietoris Sequence:

Theorem 4.5. *Suppose $A, B \subset X$ such that the union of the interiors of A, B is X. Then the following sequence is exact:*

$$\cdots \xrightarrow{\partial} H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots \longrightarrow H_0(X) \longrightarrow 0$$

4.5 Application

We'll use all these tools to show that $\mathbb{R}^m, \mathbb{R}^n$ are not homotopy equivalent for $n \neq m$. First off we'll compute the homology of the n-sphere $\mathbb{S}^n$.

Theorem 4.6. *The homology of the n-sphere is as follows:*

$$H_i(S_n) \cong \begin{cases} \mathbb{Z} & i = 0, n > 0 \\ \mathbb{Z} & i = n > 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = n = 0 \\ 0 & else \end{cases}$$

Proof. The computation for the $n = 0$ case follows from the fact that S_n has two path-connected components if $n = 0$, and 1 path connected component otherwise. We now use the Mayer-Vietoris Sequence. Let $X = \mathbb{S}^n$ and let A, B be the two hemispheres of $\mathbb{S}^n$. Then $A \cap B = \mathbb{S}^{n-1}$. Furthermore, as A, B are contractible to a point

$$H_n(A) \oplus H_n(B) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 0 \\ 0 & else \end{cases}$$

Due to this for $i \geq 1$ we get that

$$0 \to H_{i+1}(\mathbb{S}^n) \to H_i(\mathbb{S}^{n-1}) \to 0$$

is exact. Thus $H_{i+1}(\mathbb{S}^n) \cong H_i(\mathbb{S}^{n-1})$. Inductively this means

$$H_i(\mathbb{S}^n) \cong H_1(\mathbb{S}^{n+1-i})$$

Thus we just need to evaluate the first homology group. The Mayer-Vietoris sequence gives us that

$$0 \to H_1(\mathbb{S}^n) \to H_0(\mathbb{S}^{n-1}) \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(\mathbb{S}^n) \to 0$$

is exact. If $n > 1$, this is equivalent to

$$0 \to H_1(\mathbb{S}^n) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

which implies $H_1(\mathbb{S}^n) = 0$ as $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$ is exact. If $n = 1$, then we get that

$$0 \to H_1(\mathbb{S}^1) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

is exact which implies $H_1(\mathbb{S}^1) \cong \mathbb{Z}$. Lastly, $H_1(\mathbb{S}^0) \cong 0$. Combining everything gives us the desired result. $\square$

Notice that this implies that $\mathbb{S}^n$ and $\mathbb{S}^m$ are not homotopy equivalent. If we write out the long exact sequence of relative homology for $A = \mathbb{S}^n, B = \mathbb{S}^m$ for $n > m$ we can see that

$$H_i(\mathbb{S}^n, \mathbb{S}^m) \neq 0$$

for large enough n. By excision,

$$H_i(\mathbb{R}^n, \mathbb{R}^m) \cong H(\mathbb{R}^n - \{0\}, \mathbb{R}^m - \{0\}) \cong H(\mathbb{S}^n, \mathbb{S}^m) \neq 0$$

If we write out the long exact sequence of relative homology for $\mathbb{R}^n, \mathbb{R}^m$, we can now see that we cannot possibly have

$$H_i(\mathbb{R}^n) \cong H_i(\mathbb{R}^m)$$

which means they are not homotopy equivalent.