# Euler Characteristics, Gauss-Bonnett, and Index Theorems

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**Keywords** Euler Characteristics, Gauss-Bonnett, Index Theorems.

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## 1 Foreword

Within this course thus far, we have become rather familiar with introductory differential geometry, complexes, and certain applications of complexes. Unfortunately, however, many of the geometries we have been discussing remain
fairly arbitrary and, so far, unrelated, despite their underlying and inherent similarities. As a result of this discussion, we hope to unearth some of these relations and make the ties between complexes in applied topology more apparent.

2 Euler Characteristics

2.1 Introduction

In mathematics we often find ourselves concerned with the simple task of counting; e.g., cardinality, dimension, etcetera. As one might expect, with differential geometry the story is no different.

2.2 Betti Numbers

2.2.1 Chains and Boundary Operators

Within differential geometry, we count using quantities known as Betti numbers, which can easily be related to the number of \( n \)-simplexes in a complex, as we will see in the subsequent discussion. Now, before we define Betti numbers, we begin by considering an arbitrary finite simplicial complex, \( K \). As a reminder,

**Definition (simplicial complex).** A simplicial complex \( K \) is a finite set of simplices in some Euclidean space \( \mathbb{R}^m \), where an \( m \)-dimensional simplex is the convex hull of \( m + 1 \) affinely independent points, such that,

1. if \( \sigma \) is a simplex of \( K \) and \( \tau \) is the boundary of \( \sigma \), then \( \tau \) is a simplex of \( K \);
2. if \( \sigma \) and \( \tau \) are simplices, then \( \sigma \cap \tau \) is either empty or a common boundary of \( \sigma \) and \( \tau \).

Consequently, the dimension of \( K \) is the maximum of the dimensions of its simplices.

With this fresh in our minds, we can now define a simplicial \( k \)-chain.

**Definition (simplicial \( k \)-chain).** Given \( K \), a finite simplicial complex, a simplicial \( k \)-chain is a formal sum of the form \( \sum_i a_i \sigma_i \) over the oriented \( k \)-simplices \( \sigma_i \) in \( K \), with coefficients \( a_i \in \mathbb{R} \).

In other words, a simplicial \( k \)-chain is a vector whose entries are indexed by the oriented \( k \)-simplices of \( K \). Consequently, we can then define a vector space of simplicial \( k \)-chains, denoted \( C_k(K) \). Clearly, the dimension of this vector space will simply be the number of \( k \)-simplices of \( K \). Using this, we can then consider the boundary operator, \( \partial_k : C_k(K) \to C_{k-1}(K) \) defined as follows.

**Definition (boundary operator).** For a single \( k \)-complex \( \sigma = (v_{i_1} \cdots v_{i_k}) \), \( k > 0 \), construct,

\[
\partial_k : C_k(K) \to C_{k-1}(K); \quad \sigma \mapsto \sum_{n=0}^{k} (-1)^n (v_{i_0} \cdots \hat{v}_{i_n} \cdots v_{i_k}),
\]

and require \( \partial_k \) to be linearly extended, i.e., \( \partial_k \left( \sum_i a_i \sigma_i \right) = \sum_i a_i \partial_k \sigma_i \).

For consistency, we also construct \( C_{-1}(K) \) such that \( C_{-1}(K) = 0 \) and make \( \partial_0 : C_0(K) \to C_{-1}(K) \) the zero-map. As an example, to help cement these concepts, consider the following figure.
As should be evident, this simplicial complex contains a single two chain. In particular,
\[
\text{two-chain } \gamma = \langle v_1v_4v_2 \rangle + \langle v_2v_4v_5 \rangle + \langle v_2v_5v_3 \rangle + \langle v_3v_5v_6 \rangle + \langle v_1v_3v_6 \rangle + \langle v_1v_6v_4 \rangle,
\]
where our coefficients are taken to be unity for simplicity. Using our boundary operator, we then have,
\[
\partial_2 \gamma = (\langle v_4v_2 \rangle - \langle v_1v_4 \rangle + \langle v_4v_5 \rangle) + (\langle v_5v_6 \rangle - \langle v_2v_5 \rangle + \langle v_2v_4 \rangle) + (\langle v_5v_3 \rangle - \langle v_2v_3 \rangle + \langle v_2v_5 \rangle) + (\langle v_3v_6 \rangle - \langle v_1v_3 \rangle + \langle v_1v_5 \rangle) + (\langle v_6v_4 \rangle - \langle v_1v_4 \rangle + \langle v_1v_6 \rangle)
\]
\[
= \langle v_1v_4 \rangle + \langle v_5v_6 \rangle - \langle v_1v_2 \rangle - \langle v_2v_5 \rangle + \langle v_1v_3 \rangle
\]
\[
= (\langle v_4v_5 \rangle + \langle v_5v_6 \rangle - \langle v_4v_6 \rangle) - (\langle v_1v_2 \rangle + \langle v_2v_5 \rangle - \langle v_1v_3 \rangle)
\]
\[
= \alpha - \beta
\]
where,
\[
\alpha = \langle v_4v_5 \rangle + \langle v_5v_6 \rangle - \langle v_4v_6 \rangle \quad \text{and} \quad \beta = \langle v_1v_2 \rangle + \langle v_2v_5 \rangle - \langle v_1v_3 \rangle.
\]
Furthermore, since \( \partial_1 \alpha = 0 \) and \( \partial_1 \beta = 0 \), we then realize that, \( \partial_1 \partial_2 \gamma = 0 \), which is actually the result of a more general theorem: \( \partial_k \partial_{k+1} = 0 \). Excellent. At this point, we can finally start defining surfaces.

### 2.2.2 Cycles and Homology

Using the boundary operator, we can now construct vector spaces of simplicial \( k \)-cycles, i.e., \( Z_k(K) = \ker(\partial_k) \). Furthermore, we can similarly define a vector space of simplicial \( k \)-boundaries, i.e., \( B_k(K) = \text{im}(\partial_{k+1}) \). Clearly, since the boundary of a boundary is zero (as seen through \( \partial_k \partial_{k+1} = 0 \)), \( B_k(K) \) is a subspace of \( Z_k(K) \). In particular, if we considered the quotient vector space, \( H_k(K) = Z_k(K)/B_k(K) \), we can construct an important vector space known as the \( k \)th homology vector space of \( K \). As an explanation, we say that two \( k \)-cycles \( \alpha \) and \( \beta \) are \( k \)-homologous if their difference equals a \( k \)-boundary, i.e., if there is a \( (k+1) \)-chain \( \gamma \), such that \( \alpha - \beta = \partial_{k+1} \gamma \) (see Figure One). Moreover, for any \( \alpha \in Z_k(K) \), the homology class of \( \alpha \), \([\alpha]\), is the set of every \( \beta \in Z_k(K) \) such that \( \alpha \) and \( \beta \) are \( k \)-homologous. Using this \( k \)th homology vector space, we can then define a Betti number.

**Definition (Betti number).** We say that the \( k \)th Betti number of the simplicial complex \( K \), denoted by \( \beta_k(K) \), is the dimension of \( H_k(K) \). In particular,
\[
\beta_k(K) = \dim(H_k(K)) = \dim(Z_k(K)) - \dim(B_k(K)).
\]
Remark. With simplicial complexes, we are often considering said complexes over the field $\mathbb{Q}$. If we instead consider the field corresponding to the reals, i.e., $\mathbb{R}$, then this discussion can be extended to homology groups.

Using this definition, we can now examine the previous example.

\[
\begin{align*}
  k = 0 : & \quad \dim(Z_0(K)) = \dim(\ker(\partial_0)) = 6, \quad \dim(B_0(K)) = \dim(\im(\partial_1)) = 5 \Rightarrow \beta_0 = 1; \\
  k = 1 : & \quad \beta_1 = 1; \\
  k > 1 : & \quad \beta_k = 0,
\end{align*}
\]

(6) where the vector space computations for $k = 1$ and $k > 1$ have been omitted due to their fairly immense nature.

As for another example, consider the torus. Interestingly enough, because $T^n = S^1 \times \cdots \times S^1$, we simply have, $\beta_k = n$ choose $k$, seeing as $B_k = 0$ for every $k \in \mathbb{N}$ and $Z_k = n$ choose $k$, i.e., $k$ products of $S^1$ out of $n$.

2.3 Euler Characteristics

Equipped with these Betti numbers, we can now move on to the classification of complexes using Euler Characteristics.

**Theorem.** If $K$ and $L$ are simplicial complexes with homotopy equivalent underlying spaces, then the $i$th homology vectors spaces of $K$ and $L$ are isomorphic. In particular,

\[ \beta_k(K) = \beta_k(L), \text{ for every } k. \]  

(9) In other words, the Betti numbers are homotopy invariants.

**Definition (Euler Characteristic).** Let $K$ be a $k$-dimensional simplicial complex; then, the Euler characteristic is,

\[ \chi(K) = \sum_{n=0}^{k} (-1)^n \beta_n(K). \]

(10) Yet, we can also construct the Euler Characteristic from the alternating sum of the number of $k$-dimensional simplices:

\[ \chi(K) = \sum_{n=0}^{k} (-1)^n \dim(C_k(K)). \]

(11) While these may seem to be disparate definition, one can easily show that the two are equivalent. In particular,

\[ \beta_i(K) = \dim(H_i(K)) = \dim(\ker \partial_i) - \dim(\im(\partial_{i+1})) = \dim(C_i(K)) - \dim(\im(\partial_i)) - \dim(\im(\partial_{i+1})), \]

(12) so, since \( \sum_{n=0}^{k} (-1)^n (\dim(\im(\partial_i)) + \dim(\im(\partial_{i+1}))) = \dim(\im(\partial_0)) + (-1)^k \dim(\im(\partial_{k+1})) = 0, \)

\[
\sum_{n=0}^{k} (-1)^n \beta_n(K) = \sum_{n=0}^{k} (-1)^n (\dim(C_n(K)) - \dim(\im(\partial_n)) - \dim(\im(\partial_{n+1}))) \rightleftharpoons \sum_{n=0}^{k} (-1)^n \dim(C_n(K)).
\]

(13)

3 Gauss-Bonnet Theorem

3.1 Background

At this point, we aspire to see what we can come to understand about any given surface provided the Euler characteristic, or perhaps other means for determining the Euler characteristic. In order to do this, we need to first conceptualize two quantities, which play critical roles in differential geometry: Gaussian curvature and geodesic curvature. Unfortunately, these concepts are deeply rooted in differential geometry, which thus forces us to first define many other quantities.
Definition (Fundamental Form [second]). Let \( f : M \to \overline{M} \) be a differentiable immersion (injective derivative) of a manifold \( M \) of dimension \( n \) into a Riemannian manifold \( \overline{M} \) of dimension \( k = n + m \). Furthermore, construct \( B(X,Y) \) for \( X,Y \) local vector fields on \( M \) such that
\[
B(X,Y) = \overline{\nabla}_X Y - \nabla_X Y,
\]
where \( \nabla \) and \( \overline{\nabla} \) are the covariant derivatives of \( M \) and \( \overline{M} \), respectively. Our second fundamental form of \( f \) at \( p \) along the normal vector \( \eta \) is then simply,
\[
\II_\eta(x) = \langle B(x,x), \eta \rangle \quad \text{for } x,y \in T_pM.
\]

Definition (Principal Curvatures). Let \( M \) be a Riemannian manifold, \( p \in M \) and \( \eta \in (T_pM)^\perp \) a unit normal vector. For \( f : M \to \overline{M}^{n+1} \) the immersion above, the principal curvatures of \( M \) are the real eigenvalues \( \lambda_1, \ldots, \lambda_n \) of the linear Hermitian operator \( S_\eta : T_pM \to T_pM \) given by,
\[
\langle S_\eta(x), y \rangle = \langle B(x,y), \eta \rangle.
\]

Definition (Gaussian curvature). Let \( M \) be a Riemannian manifold, \( p \in M \) and \( \eta \in (T_pM)^\perp \) a unit normal vector. For \( f : M \to \overline{M}^{n+1} \) the immersion above, the Gaussian curvature, \( K \), of \( f \) at \( p \) is given by,
\[
K = \det(S_\eta) = \lambda_1 \cdots \lambda_n.
\]

Remark. For our purposes, we will often define the Gaussian curvature as \( K = \det(DN) \), where \( \gamma : M \to S^2 \) is the Gauss map, i.e., the continuous map such that \( N(p) \) is a unit vector orthogonal to \( M \) at \( p \in M \).

Definition (Geodesic curvature). Let \( M \) be a Riemannian manifold and \( \gamma : I \to M \) an arbitrary curve; then the geodesic curvature is the function, \( \kappa_g : I \to \mathbb{R} \), given by,
\[
k_g(s) = ||\nabla_{\gamma'(s)} \gamma'(s)||.
\]

Wonderful. We can finally introduce the Gauss-Bonnet theorem and then give some meaning to these quantities.

Theorem (Gauss-Bonnet). Let \( M \) be a compact two-dimensional Riemannian manifold with boundary \( \partial M \). Let \( K \) be the Gaussian curvature of \( M \), and let \( k_g \) be the geodesic curvature of \( \partial M \); then,
\[
\int_M K \, dA + \int_{\partial M} k_g \, ds = 2\pi \chi(M),
\]
where \( \chi(M) \) is the Euler characteristic of \( M \).

3.2 Examples

Consider the example of a spherical triangle, i.e., a triangle on a sphere of radius \( R \).
Figure 2: A geodesic triangle on $S^2$.

By what we know from earlier, if $M = \triangle \alpha \beta \gamma$, then $\chi(M) = 1$. Furthermore, since $K_{S^2} = 1/R^2$,

$$\int_M K \, dA = (1/R^2) \int_M dA = A/R^2,$$

where $A$ is the area of our triangle. Thus, by the Gauss-Bonnet theorem,

$$A/R^2 = 2\pi - \int_{\partial M} k_g \, ds.$$  \hspace{1cm} (20)

However, how can we compute the geodesic curvature component? Well, since the edges of our triangles are geodesics, we need only worry about the vertices. In particular,

$$\int_{\partial M} k_g \, ds = (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma),$$ \hspace{1cm} (21)

seeing as these are the turning angles at each vertex. Consequently,

$$A = R^2(\alpha + \beta + \gamma - \pi) \quad \text{or} \quad \alpha + \beta + \gamma = \pi + A/R^2.$$ \hspace{1cm} (22)

For a more complex example, we can consider four-dimensional spacetime, in which case the Gauss-Bonnet theorem for a compact oriented manifold gives,

$$\chi(M) = \frac{1}{32\pi^2} \int_M d^4x \sqrt{|g|} E(R^2 - 4|Ric|^2 + |Riem|^2).$$ \hspace{1cm} (23)

By examining this closely, one finds that this is the second-order curvature extension of the Einstein-Hilbert action,

$$S = \frac{c^4}{8\pi G} \int d^4x \sqrt{-g} R.$$ \hspace{1cm} (24)

Yet, because this Gauss-Bonnet term is a topological invariant in four dimensions, i.e., equivalent to the constant $\chi(M)$, this implies that by adding such a term to the action, the equations of motion will remain unchanged! Wow!

4 \hspace{1cm} Index Theorems

The goal of this section will be to explain and prove the Poincare-Hopf theorem, which relates the euler characteristic to some nice local properties. Using this we will gain some nice corollaries such as the Hairy Ball Theorem. For everything we discuss in this section assume $M, N$ are smooth manifolds and that $f : M \to N$ is a smooth map.
4.1 The Differential of a Manifold Map

One way of viewing $T_x M$ for a point $x \in M$ is as an equivalence class of smooth curves $\gamma : (-1, 1) \to M$ with $\gamma(0) = x$. We say that two curves are “equivalent” if $\gamma'(0)$ is the same tangent vector. This definition is nice because it actually gives us a way of mapping $T_x M \to T_{f(x)} N$ by mapping $\gamma$ to another curve. The differential is precisely the map we get by doing this. To be specific:

**Definition (Differential).** Let $f : M \to N$ be a smooth map. For all $x \in M$, $f$ induces a differential map $df_x : T_x M \to T_{f(x)} N$ such that if $\gamma : (-1, 1) \to M$ is a smooth curve with $\gamma(0) = x$ then:

$$df_x(\gamma'(0)) = (f \circ \gamma)'(0)$$

This definition has a nice consequence in that it allows us to actually give $T_x M$ the structure of a vector space!! As $M$ is a manifold, there is some map $\phi : U \to \mathbb{R}^n$ where $U$ is a chart containing $x$. Notice that the values $d\phi_x(\gamma'(0))$ is actually now a vector in $\mathbb{R}^n$, which allows us to add them as vectors in $\mathbb{R}^n$ and multiply them by scalars. In fact, under this identification $df_x$ becomes a linear map between vector spaces.

Before we can define the degree and index we need one last definition:

**Definition (Regular Point).** Let $f : M \to N$ be a smooth map and let $p \in M$ be a point. We say $p$ is a regular point of $f$ if the differential map $df_p : T_p M \to T_{f(p)} N$ is surjective. Furthermore, $y \in N$ is a regular value if all the points in $f^{-1}(y)$ are regular points.

This definition should actually be pretty familiar from calculus. If you take $M, N$ to both be $\mathbb{R}$, then the regular points are just those points $P$ that aren’t critical values (i.e. where $f'$ is not zero).

4.2 Degree and Index of a Manifold Map

4.2.1 Technical Definition

**Definition (Degree).** Let $f : M \to N$ be a smooth map of manifolds of the same dimension. Suppose $y \in N$ is a regular value. Let $\text{sign}(df_x)$ be the sign of $\text{det}(df_x)$ as a linear map (i.e. 1 if the determinant is positive and -1 if the determinant is negative). Then we define the degree of $f$ at $y$ to be:

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign}(df_x)$$

**Remark.** The reason regular value is important here is so that $\text{det}(df_x)$ is defined and nonzero. As $df_x$ is surjective and $T_x M, T_{f(p)} N$ have the same dimension it is an isomorphism of vector spaces. This means as a linear map it has nonzero determinant.

**Remark.** Degree is actually independent of choice of regular value. In particular it is actually homotopy invariant. We’re not going to prove this here, but it’s important as we actually just write $\deg(f)$ instead of $\deg(f, y)$. This is important for our next definition.

**Remark.** To use Wikipedia’s words the degree “is a number that represents the number of times that the domain manifold wraps around the range manifold under the mapping”
**Definition (Isolated Zero).** Let $M$ be a compact manifold and let $V$ be a vector field on $M$. We say $p \in M$ is an isolated zero of $V$ if $V(p) = 0$ and there exists an open region $U$ containing $p$ such that for all $q \in V$ distinct from $p$ we have $V(p) \neq 0$.

**Definition (Index of a point).** Let $M$ be a compact manifold and let $V$ be a vector field on $M$. Suppose $p \in M$ is an isolated zero of $V$. Let $\phi : U \to \mathbb{R}^n$ be some chart around $p$ and let $S^n_{\epsilon}$ be some ball around $x$ in $U$ small enough that $V$ has no other zeroes in $S^n_{\epsilon}$. Lastly, let $\hat{v} : S^n_{\epsilon} \to S^n_{\epsilon}$ be given by:

$$
\hat{v}(x) = \frac{V(x)}{|V(x)|}
$$

Then we can define the index as:

$$
\text{Ind}(V, p) = \deg(\hat{v})
$$

### 4.2.2 A More Concrete Case

In the case of a 2-manifold, we can also define the index using a line integral:

$$
\text{Ind}(V, p) = \frac{1}{2\pi} \int_{\gamma} d\theta_V = \frac{1}{2\pi} \int_{\gamma} \frac{v_x dy - v_y dx}{v_x^2 + v_y^2}
$$

where our vector field $V = (v_x, v_y)$. Here $\gamma$ is the same $S^{2-1}_{\epsilon}$ ball that we took in the definition above. Notice that $\theta_V$ is the angle $V$ in local coordinates. From this we can view $\text{Ind}(V, p)$ as the number of turns $V$ makes in the small curve around $p$.

### 4.3 The Poincare-Hopf Theorem and Applications

We can now give the Poincare-Hopf Theorem:

**Theorem (Poincare-Hopf Theorem).** Let $M$ be a compact manifold and let $V$ be a vector field on $M$ such that $V$ has isolated zeros. Let $\text{Fix}(V)$ be the set of $p \in M$ such that $V$ vanishes at $p$. Then

$$
\sum_{p \in \text{Fix}(V)} \text{Ind}(V, p) = \chi(M)
$$

This theorem is really powerful in that it holds for all vector fields on $M$ and allows one to recover the euler characteristic (a fact about the entire manifold) from just a tiny region around the singularities of the vector field.

In fact, the hairy ball theorem is actually a direct corollary of the Poincare-Hopf Theorem:

**Corollary (Hairy Ball Theorem).** Let $M$ be a compact manifold. If $\chi(M) \neq 0$, then $M$ has no nonvanishing vector fields.

**Proof.** Suppose $M$ had a nonvanishing vector field $V$. Then $\text{Fix}(V)$ is empty. Thus

$$
\sum_{p \in \text{Fix}(V)} \text{Ind}(V, p) = 0
$$

vacuously. However, as $\chi(M) \neq 0$ this is a contradiction on Poincare-Hopf so we cannot have any nonvanishing vector fields.

In particular, the usual sphere has euler characteristic 2 so it has no nonvanishing vector fields! We now give one other result:
Corollary. Any rotation of the 2-sphere has 2 fixed points.

Proof. We will assign a vector field $V$ as follows. For each point $p$ on the 2-sphere, let $p'$ be the image of $p$ under rotation. The vector we assign to $p$ is the one that lies along the great circle connecting $p, p'$ with magnitude $||p - p'||$. By the properties of rotation, this will be smooth. Note that for this vector field $V$, $\text{Fix}(V)$ is precisely the set of fixed points of the rotation. Now for each $p \in \text{Fix}(V)$, we can see that

$$|\text{Ind}(V, p)| = 1$$

As the euler characteristic of the 2-sphere is 2, in order to have

$$\sum_{p \in \text{Fix}(V)} \text{Ind}(V, p) = 2$$

we need $|\text{Fix}(V)| \geq 2$. \qed

References