1 Introduction

We’ve talked a lot about Ho’s (Homeomorphism, Homotopy, Homology) this semester, and this week is going to be knotty.

1.1 What is a Knot?

Topology is the study of geometric objects and how they are preserved under deformations. Knot theory is just one of the sub-fields of topology. Now what are knots? Just think of a knotted loop of string, with no thickness, and it’s cross-section being a single point that connects the ends of the string. A knot is a closed curve in space that does not intersect itself anywhere and a link is a set of knotted loops tangled up together (like 3 rings intertwined).

![Figure 1: How to form a knotted loop from a piece of string](image1)

Deformed curves are considered to be the same knot. So, deforming a knot does not change it’s state.

![Figure 2: All the same knot](image2)

There are different types of knots. The simplest knot to form is the unknotted circle. We’ve seen this in figure 1, before cutting the string. This type of knot is called a unknot or trivial knot. The next simplest knot is one with three loops intertwined. This knot is called a trefoil knot.
But how can we tell if two knots are distinct? Simple. If you construct one knot out of string, try to deform it into the shape of the other knot. If you can, they are the same knot. If not (pun intended), the two knots are distinct. Let’s look at another trefoil knot.

Now you might be wondering how this knot is a trefoil knot. It doesn’t look anything like the previous image of a trefoil knot. Well let’s remember how two knots are distinct. If you stare at it long enough, you can see how you can deform this into the default view of a trefoil knot. But a more fun way to approach this is to try it out with some string.

1.2 Knot Properties

Now that we know there are many different ways to show a type of knot, let’s delve deeper into the knot properties. Below is a new type of knot called a **figure-eight knot**.

The above knots are called **projections** of the figure-eight knot. Projections are knots that can always be deformed to the default view of a type of knot. For example, figure 4 is a presentation of the trefoil knot. Now let’s go back to the projections of a figure-eight knot. The places where the knot crosses itself or where the string overlaps is called the **crossings** of a projection. The three projections of the figure-eight knot each have four crossings so we can say that there are no projections of the figure-eight knot with fewer than four crossings. Now we can more formally define a nontrivial knot.
A knot is nontrivial if it has more than one crossing in its projection. Let’s recall what a trivial knot looks like (figure 3) and compare it with the below examples.

![Figure 6: Again...all the same knot](image)

Any of the above projections can be deformed to look like the trivial knot (untwist the crossing). In addition, any of the projections can be made to look like the other without undoing the crossing.

### 1.3 Composition of Knots

Now let’s discuss how we build new knots. Using two projections of knots denoted J and K, you can create a new knot by cutting an arc from each projection and then use that cutout to connect the two projections. The new knot formed is then called the composition of the two knots, denoted JK. Let’s look at an example.

![Figure 7: A correct composition](image)

We can’t just cut anywhere we want. Let’s talk about the prerequisites for knot compositions:

1) The two projections used can’t overlap
2) The arcs being cut out must be on the outside of each projection
3) When selecting an arc to cut, avoid any crossings on the projection

Now we know how to create a proper knot composition. Let’s see what happens when we don’t follow the rules.

![Figure 8: An incorrect composition](image)
Now let’s discuss different types on compositions. A composition of two knots where neither of the projections are trivial knots is called a **composite knot** and the knots that form this composite knot are called **factor knots**. But what happens when one of the projections is a trivial knot? When combining knots, think of the trivial knot as the identity element. Let’s look at an example.

![Figure 9: Prime Knot](image)

This should make sense conceptually. Combining a knot with the trivial knot is simply extending the length of the arc on first knot. It doesn’t create a distinct knot. A **prime knot** is a composition not formed from two nontrivial knots. But what about the trivial knot? Is it composite? No, because there is no way to take the composition of two nontrivial knots to form the trivial knot. This is analogous to the prime factorization of numbers in that a composite knot factors into a set of prime knots.

We can also add **orientation** to a knot by choosing a direction to travel along the knot. The composition of two oriented knots have two distinct possibilities, the orientation of the two projections are the same or they are different. All of the compositions where the orientation of the projections are the same will yield the same composite knot and all of the compositions where the orientation of the projections are different will yield the same composite knot. Let’s look at an example.

![Figure 10: Oriented Knots](image)

Note that since projections of the composite knots \(a\) and \(b\) have the same orientation, the composite knots are the same. But what about composite knot \(c\)? Actually, composite knot \(c\) is the same as the other two. Why? Because one of it’s factor knots is **invertible**. A knot is invertible if it can be deformed back to itself while reversing the orientation. It’s difficult to visualize but you can invert the left factor knot in composite knot \(c\) so the orientation coincides with the other composite knots.

### 1.4 Reidemeister Moves

Planar isotopy is a deformation within the same projection plane. A **Reidemeister move** is one of three ways to deform a projection of a knot that will change the crossings of the projection. Now what are the Reidemeister moves:
1) Put in or take out a loop

![Figure 11: Reidemeister Move 1](image)
The knot type should remain unchanged after this move.

2) Add two crossings or remove two crossings

![Figure 12: Reidemeister Move 2](image)
Remember a crossing is simply one part of the string overlapping another part.

3) Slide a strand of the projection from one side of a crossing to the other side of the crossing.

![Figure 13: Reidemeister Move 3](image)
Note that each of these moves changes the projection of the knot but does not change the actual knot represented by the projection. These moves are each called an **ambient isotopy**.
How about some practice. Can you find the Reidemeister moves to get from the following projection to the trefoil knot?

Figure 14: Practice Example 1

Figure 15: Collection of prime knots up to 7 crossings. The ordering are traditional dating back to Tait (1898).
2 Brackets and Polynomials

Polynomials are one the most successful and interesting ways to tell knots apart. One can compute the polynomial from a knot projection, with the key feature that any two projections of the same knot yield the same polynomial. The Jones Polynomial is a Laurent polynomial (terms can take both positive and negative exponents) that is invariant under all three Reidemeister moves. It is a necessary, but no sufficient, condition for showing two knots are the same.

2.1 Kauffman Bracket

Kurt Kauffman came up with the so-called bracket polynomial in the 1920s. The Kauffman Bracket associates each a knot (as a 2D projection) with an algebraic expression (a Laurent polynomial). It can be applied to either knots or links \( \langle K \rangle \). It works by starting with the knot \( K \) and iteratively smoothing each crossing with algebraic rules, until we get to the unknot. Since there are 2 resolutions for each crossing, the Bracket polynomial with have \( 2^n \) expressions before collecting terms. Fortunately a lot of these are repeated, as we shall see.

We can formulate the Kauffman Bracket with three rules:

1. Let us assign the unknot identity \( a^0 = 1 \).

\[
\text{Rule 1: } \langle \bigcirc \rangle = 1
\]

2. Second we prescribe how to resolve and smooth crossings. We can decrement the number of crossings in a knot by using the Skein Relations Anywhere there is an ‘X’ crossing, we can open the X into vertical

\[
\text{Rule 2: } \begin{align*}
\langle X \rangle &= A \langle \rangle + A^{-1} \langle \bigotimes \rangle \\
\langle X \rangle &= A \langle \bigotimes \rangle + A^{-1} \langle \bigotimes \bigotimes \rangle
\end{align*}
\]

and horizontal resolutions. One gets a factor of \( A \) and the other \( A^{-1} \). Although there appear to be two relations, the reader can verify that they are in fact the same expression rotated by 90 degrees.

3. The third rule deals with removing an unknot from a partially untangled knot

\[
\text{Rule 3: } \langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle
\]

Let’s practice this with some examples. For two unlinked circles:

\[
\langle \bigotimes \rangle = A \langle \bigotimes \rangle + A^{-1} \langle \bigotimes \rangle
\]

\[
= A (A \langle \bigotimes \rangle + A^{-1} \langle \bigotimes \bigotimes \rangle) + A^{-1} (A \langle \bigotimes \rangle + A^{-1} \langle \bigotimes \bigotimes \rangle)
\]

\[
= A (A(-A^2 + A^{-2})) + A^{-1} (1) + A^{-1} (A(1) + A^{-1} (-A^2 + A^{-2}))
\]

\[
= -A^4 - A^{-4}
\]

Notice how on the path to resolution Skein Relations will make links out of knots, as well as knots out links.

\[\text{One example pair which Jones Polynomial fails to distinguish is } (5_1, 10_{132})\]
2.2 Invariance under Reidemeister Moves

Two knots are the same, if one can be continuously transformed to the other with the help of Reidemeister moves. For the Kauffman Bracket to correctly distinguish knots, it must also survive the Reidemeister moves.

![Diagram of three Reidemeister moves](image)

Figure 16: The three Reidemeister moves preserve knot invariance

- **Ω₂ works**

\[
<\frown> = A<\frown> + B<\frown>
\]

\[
= A(A<\frown> + B<\frown>) + B(A<\frown> + B<\frown>)
\]

\[
= A(A<\frown> + BC<\frown>) + B(A<\frown> + B<\frown>)
\]

\[
= (A^2 + ABC + B^2)<\frown> + BA<\frown> (\text{after rearranging})
\]

The second Reidemeister move works. This calculation imposes \( B = A^{-1} \) for the Bracket Polynomial.

- **Ω₃ works**

\[
<\multimap> = A <\multimap> + A^{-1} <\frown>
\]

(Now, apply the fact that Type II moves don’t change the bracket polynomial)

\[
= A <\multimap> + A^{-1} <\frown> = <\multimap>
\]

The third Reidemeister move works, and the calculation is considerably simplified by leveraging invariance under Ω₂.

- **Ω₁ runs into a problem**

\[
<\overleftarrow{\bowtie}> = A<\overleftarrow{\bowtie}> + A^{-1} <\frown>
\]

\[
= A(-A^2 - A^{-2})<\overleftarrow{\bowtie}> + A^{-1} <\overleftarrow{\bowtie}>
\]

\[
= -A^3<\overleftarrow{\bowtie}>
\]

\[
<\overleftarrow{\circlearrowleft}> = A<\overleftarrow{\circlearrowleft}> + A^{-1} <\overleftarrow{\circlearrowleft}>
\]

\[
= A<\overleftarrow{\circlearrowleft}> + A^{-1}(-A^2 - A^{-2})<\overleftarrow{\circlearrowleft}>
\]

\[
= -A^3<\overleftarrow{\circlearrowleft}>
\]

Each untwisting of the loop appears to pick up a factor of \( A^3 \).

The Kauffman Bracket preserves 2 out 3 Reidemeister moves, but fails to preserve the twist in Ω₁. Every loop from Ω₁ introduces a pesky factor of \( A^3 \).
2.3 Writhe and Jones Polynomial

To remedy this we define the concept of writhe:

Orientation is introduced to each knot by picking a direction and tracing that arrow through every crossing. Then a crossing can be distinguished between right handed (+) and left handed (−).

The sum of all such crossings is defined to be writhe. Recognize that each transformation by Reidemeister move $\Omega_1$ increments the total writhe by 1! We can cancel this effect by introducing a factor of $-A^{-3}$ for each degree of writhe in the link projection

$$X(L) = (-A^{-3})^{w(L)} \langle L \rangle$$

To arrive at the Jones Polynomial, we simply collect terms $A^4 \equiv q$. 

$$X(L') = (-A^3)^{-w(L')} \langle L' \rangle$$
$$= (-A^3)^{-w(L)+1} \langle L' \rangle$$
$$= (-A^3)^{-w(L)+1} ((-A)^3 \langle L \rangle)$$
$$= (-A^3)^{-w(L)} \langle L \rangle = X(L)$$
As an exercise let us compute the Jones Polynomial for a trefoil knot: First we start with the bracket

$$
\langle \begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array} \rangle = A\langle \begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array} \rangle + A^{-1}\langle \begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array} \rangle \\
= A(-A^4 - A^{-4}) + A^{-7} \\
= (A^{-7} - A^{-3} - A^5).
$$

Next we should compute the writhe. We see all the crossings are either $1 + 1 + 1 = 3$ or $-1 - 1 - 1 = -3$. So let us take the right-handed orientation and find

$$
X(L) = (-A^{-3})w(L) \langle L \rangle \\
= (-A^{-3})^3(A^{-7} - A^{-3} - A^5) \\
= A^{-9}(-A^{-7} + A^{-3} + A^5) \\
= -A^{-16} + A^{-12} + A^{-4} \\
= -q^{-4} + q^{-3} + q^{-1}.
$$

To confirm here is a table of Jones Polynomials for knots up to 7 crossings

| 3_1 | -1 | 1 | 0 | 1 | 0 |
| 4_1 | 1 | -1 | 1 | -1 | 1 |
| 5_1 | -1 | 1 | 1 | -1 | 0 | 0 | 0 |
| 5_2 | -1 | 1 | 1 | -1 | 2 | -1 | 1 | 0 |
| 6_1 | 1 | -1 | 1 | -2 | 2 | -1 | 1 |
| 6_2 | 1 | -2 | 2 | -2 | 2 | -1 | 1 |
| 6_3 | -1 | 2 | -2 | 3 | -2 | 2 | -1 |
| 7_1 | -1 | 1 | 1 | -1 | 1 | -1 | 0 | 0 | 0 |
| 7_2 | -1 | 1 | 1 | 2 | -2 | 2 | -1 | 1 | 0 |
| 7_3 | 0 | 0 | 1 | -1 | 2 | -2 | 3 | -2 | 1 | -1 |
| 7_4 | 0 | 1 | -2 | 3 | -2 | 3 | -2 | 1 | -1 |
| 7_5 | -1 | 2 | -3 | 3 | -3 | 3 | -2 | 1 | 0 |
| 7_6 | -1 | 2 | -3 | 4 | -3 | 3 | -2 | 1 |
| 7_7 | -1 | 3 | -3 | 4 | -4 | 3 | -2 | 1 |

where bolded numbers indicate the unknot $q^0 = 1$ constant term.

### 3 Khovanov Homology

The Khovanov Homology is the categorification of Jones Polynomial. The main idea of the Khovanov Homology is to replace the Kauffman bracket with "the Khovanov bracket" $[[L]]$, which is a chain complex of graded vector spaces whose graded Euler characteristic is just the Kauffman bracket. Thus, the Khovanov bracket can be described similarly to the Kauffman bracket:

$$
\left[ \emptyset \right] = 0 \rightarrow \mathbb{Z} \rightarrow 0; \quad \left[ \bigodot L \right] = V \otimes [L]; \quad \left[ \bigotimes \right] = F \left( 0 \rightarrow [\bigotimes] \xrightarrow{d} [\bigodot \bigodot \{1\}] \rightarrow 0 \right).
$$

Figure 19: The Khovanov Bracket
### 3.1 Revisiting the Jones Polynomial

**Definition** Let $\chi$ be the set of crossings in a link projection $L$ (of an oriented link in an oriented Euclidean space) and let $n = |\chi| = n_+ + n_-$, the number of right-handed and left-handed crossings. The **unnormalized Jones polynomial** of $L$ is $\hat{J}(L) = (-1)^nq^{n_+ - 2n_-} < L >$. This can be normalized into the Jones polynomial $J(L) = \frac{\hat{J}(L)}{q^{n_+}}$ (and substituting $q = -t^2$).

**Calculation** The Jones polynomial then can also be calculated in an alternate way. First define 0-smoothing and the 1-smoothing of a crossing respectively, based on Figure 1. Thus, for any n-crossing knot, we can create a "complete smoothing" $S_\alpha$ of $L$, for $\alpha \in \{0,1\}^\chi$. Each $\alpha$, therefore, represents a vertex in an n-dimensional cube as shown in Figure 24 for the trefoil. Let $r$ be the "height" of the crossing, or the number of 1-smoothings. Then for each smoothing, according to the unnormalized Jones polynomial, we have $(-1)^r q^{r(q + q^{-1})^k}$. Summing over these all the $\alpha$ (its an alternating sum) will give the unnormalized Jones polynomial.

Figure 20: Jones polynomial

### 3.2 Categorification

**Definition** A **graded vector space** $W = \bigoplus_m W_m$ with homogeneous components $\{W_m\}$ is a direct sum vector spaces $W_m$. An example of a grade vector space are polynomials, were each homogeneous component of degree $m$ are linear combinations of monomials of degree $m$. The **graded dimension** of $W$ $q\dim W$ is defined to be the power series $\sum_m q^m \dim W_m$.

**Definition** Let ${\cdot |l}$ be the **degree shift operation** on graded vector spaces such that $W{\cdot |l}_m = W_{m-l}$, thus $q\dim W{l} = q^l q\dim W$.
**Definition**  Similarly, let \([s]\) be the **height shift operation** on a chain complex \(\mathcal{C} = \ldots \to \mathcal{C}^r \xrightarrow{d^r} \mathcal{C}^{r+1} \to \ldots\) given by if \(\mathcal{C}[s]^r = \mathcal{C}^{r-s}\) with the differentials shifted accordingly.

Thus returning to our diagram from before, let \(V\) be a graded vector space with two basis elements \(v_\pm\) with degrees \(\pm 1\) so that \(q \dim V = q + (-1)^{-1}\). Therefore, for each vertex \(\alpha\), we associate a graded vector space \(V_\alpha(L) = V^{\oplus kr}\), where \(k\) is the number of cycles (unknots) in the smoothing of \(L\) according to \(\alpha\) and \(r\) is the height, i.e. \(r = |\alpha| = \sum \alpha_i\). With this construction, we see that for each vertex \(\alpha\) \(q \dim V_\alpha(L)\) is the polynomial that appears in the Figure 1. Finally, let the \(r^{th}\) chain group \([[L]]^r\) be the direct sum of all vector spaces with height \(r\), i.e. \([[L]]^r = \oplus_{|\alpha|=|r|} V_\alpha(L)\). Finally, set \(C(L) = [[L]]([-n_-]|n_+-2n_-)\). This is shown below in Figure 25.

![Figure 21: Graded Vector Space](image)

**3.3 Maps**

In order to make Figure 25 into a chain complex, we need to give it a differential along each edge \(\xi\). Starting from the left, we will construct \(d_\xi\) along each edge by sending a single 0 to 1 (tail to 0 and head to 1), which is denoted by a *. That is, if \(\xi = *00\), then it sends 000 to 100, corresponding to the top choice from the left. Then define \(|\xi|\) or the height of an edge \(\xi\) as the height of its tail. Then, we can collapse the diagram vertically to a complex and define \(d^r = \sum_{|\xi|=r} (-1)^{\xi} d_\xi\). \((-1)^{\xi}\) is defined to be \((-1)^{\sum_{i<j} \xi_i}\), where \(j\) is the position of * in \(\xi\). This condition is necessary to make the faces anticommute. Since it is easier to show a
Diagram which positively commutes, the negative edges in Figure 4 are shown with circles on the tails to indicate a negative.

Finally, it remains to find maps $d_\xi$ to make the cube commutative and are of degree 0. Thus, define linear maps $m, \Delta$ as follows:

\[
\begin{align*}
(\circ \circ \circ \circ \circ \circ) &\rightarrow (V \otimes V \overset{m}{\rightarrow} V) & m : &\begin{cases}
  v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+
  v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0
\end{cases} \\
(\circ \circ \circ \circ \circ \circ) &\rightarrow (V \overset{\Delta}{\rightarrow} V \otimes V) & \Delta : &\begin{cases}
  v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+
  v_- \mapsto v_- \otimes v_-
\end{cases}
\end{align*}
\]

Thus, giving us the following diagram for the trefoil. It is a fact that these maps make the diagram in Figure 26 and make $d_\xi$ have degree 0 (left to the reader as an exercise).
3.4 Theorems

**Definition** the **graded Euler characteristic** of a chain complex $C$ is the alternating sum of the graded dimensions of its homology groups. If the differential $d$ has degree 0 and the chain groups are finite dimensional, then the Euler characteristic is equal to the alternating sum of the graded dimensions of the chain groups.

**Theorem** The graded Euler characteristic of $C$ is equal to the unnormalized Jones polynomial of L: $\chi_q(C(L)) = \hat{J}(L)$

**Proof** Trivial by construction.

**Theorem** The n-dimensional cube in Figure 28 (and Figure 26) are commutative (without signs) and the sequences $[[L]]$ and $C(L)$ are chain complexes.

**Proof** A routine verification.

**Definition** Let $\mathcal{H}^r(L)$ be the $r^{th}$ cohomology of the complex $C(L)$. Let $Kh(L)$ be the graded Poincare polynomial of the complex $C(L)$ in the variable $t$, so $Kh(L) = \sum_r t^r q \dim \mathcal{H}^r(L)$.

**Theorem** The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are link invariants. Therefore, $Kh(L)$ is a polynomial in $t$ and $q$, is also a link invariant. Furthermore, $Kh(L)$ equals the unnormalized Jones polynomial when $t = -1$.

**Proof** It’s long.

**Theorem** The Khovanov homology is better than the Jones Polynomial.

**Proof** $\mathcal{H}(C(5_1)) \neq \mathcal{H}(C(10_{132}))$ but $\hat{J}(5_1) = \hat{J}(10_{132})$
Bonus 1: Real Knots

In real life, useful knots are strong but easy to undo.

1. Overhand Knot ($3^1$)
2. Figure-8 Knot ($4^1$)
3. Square Knot ($3^1 \# 3^1$)
4. Slip Knot ($1^1$)
5. Bowline Knot ($6^3$)

Bonus 2: Fish

Figure 25: Applied Topology in Biology: https://www.youtube.com/watch?v=RrPvMMkQkk0

4 References

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