Open/Closed Correspondence and Mirror Symmetry

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Open/Closed Correspondence

- Proposed in physics 20+ years ago as a class of string dualities
  - [Mayr ’01, Lerche-Mayr ’01]

  \[
  \text{open strings on Calabi-Yau 3-folds} \quad \xrightarrow{\text{genus zero}} \quad \text{closed strings on Calabi-Yau 4-folds}
  \]

- Mathematically: proposed relations between Gromov-Witten theories
Open/Closed Correspondence

• Mathematically: proposed relations between Gromov-Witten theories

\[ \text{open GW on toric CY 3-folds relative to Lagrangians} \xleftrightarrow{g=0} \text{closed GW on toric CY 4-folds} \]

• Should hold on multiple levels:
  ○ Numerical invariants at individual curve classes
  ○ Generating functions
  ○ Givental-style mirror symmetry ($J$- and $I$-functions)
  ○ B-model mirror families, periods, Picard-Fuchs systems
  ○ Wall-crossings, crepant transformations
  ○ ...
Road Map of Construction

\[ \text{Open} \quad \rightarrow \quad \text{Relative} \quad \rightarrow \quad \text{Local} \quad \rightarrow \quad \text{Closed} \]

\[(X, L) \quad \rightarrow \quad (X \cup D, D) \quad \rightarrow \quad \mathcal{O}_{X \cup D}(-D) \quad \rightarrow \quad \tilde{X} \]

\[\dim_{\mathbb{C}} = 3 \quad 3 \quad 4 \quad 4\]
Open Geometry

- $X$: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold (or orbifold)
  - E.g. $\mathbb{C}^3$

\[ \begin{align*}
\text{C coordinate axes} & \quad \text{image under moment map} \\
& \quad \text{of real CY 2-subtorus } T'_R \\
& \quad \text{toric fan}
\end{align*} \]
Open Geometry

- $X$: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold
  - Additional examples:
    - Resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{P}^1$
    - $\mathcal{O}_{\mathbb{P}^2}(-3)$

  - More general examples come from canonical bundles of toric surfaces
  - Toric fan = cone over polytope with regular triangulation
• $X$: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold

• We assume that $X$ is semi-projective, i.e. a symplectic quotient
  
  o $\exists$ Hamiltonian $U(1)^k$-action on $\mathbb{C}^{k+3}$, with moment map $\mu : \mathbb{C}^{k+3} \rightarrow \mathbb{R}^k$

  o $X = \mu^{-1}(r)/U(1)^k$, where $r$ is a Kähler class

  o Standard Kähler form on $\mathbb{C}^{k+3}$ descends to symplectic form on $X$

• Equivalently: $X$ is a GIT quotient, support of fan is convex
Aganagic-Vafa Lagrangian

- $X$: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold

- $L$: Lagrangian of Aganagic-Vafa type
  - $L = (\mu^{-1}(r) \cap \text{codim} \mathbb{R} = 3 \text{ constraint}) / U(1)^k$
  - Preserved under action of real CY 2-subtorus $T'_R$

  - Topology: non-compact, $\cong S^1 \times \mathbb{R}^2$ in smooth case

  - Intersects a unique 1-dim torus orbit - we assume this orbit is non-compact
Aganagic-Vafa Lagrangian

- $X$: $\dim_{\mathbb{C}} = 3$ toric Calabi-Yau manifold
- $L$: Lagrangian of Aganagic-Vafa type
  - $L$ bounds a disk $B$ in the torus orbit
  
  $H_1(L) = \mathbb{Z}[\partial B], \ H_2(X, L) = H_2(X) \oplus \mathbb{Z}[B]$

- $f \in \mathbb{Z}$: additional parameter called framing of $L$
Open Gromov-Witten Invariants

- Virtual counts of stable maps from bordered Riemann surfaces to $(X, L)$
  - Topological vertex, all-genus mirror symmetry, crepant transformations...
  - Current project: understand them by relating to closed invariants

- We focus on disk invariants
  - Curve class: $\beta' = \beta + d[B] \in H_2(X, L)$
  - Interior insertions: $\gamma_1, \ldots, \gamma_n \in H^2(X; \mathbb{Q})$
  - Defined by $T'_R$-localization

$$
\langle \gamma_1, \ldots, \gamma_n \rangle_{X, (L,f)}^{X, (L,f)} := \int_{\overline{M}_{0,1}, n(X, L|\beta', d)}^{T'_R \text{vir}} \frac{i^* \prod_{i=1}^n \text{ev}_i^* \gamma_i}{e_{T'_R}(N_{\vir})} \bigg|_{\text{wt restriction}} \in \mathbb{Q}
$$
From Open to Relative

- **Relative** geometry: add a new toric divisor $D \cong \mathbb{C}^2$ to $X$ depending on $(L, f)$

$$\begin{align*}
X &= \mathbb{C}^3 \\
X \sqcup D &= \mathcal{O}(f) \oplus \mathcal{O}(-f - 1)/\mathbb{P}^1
\end{align*}$$

- $(X \sqcup D, D)$ is log Calabi-Yau: $K_{X \sqcup D} = \mathcal{O}_{X \sqcup D}(-D)$

- Isomorphism $H_2(X, L) \xrightarrow{\sim} H_2(X \sqcup D), [B] \mapsto [\mathbb{P}^1]$
From Relative to Local and Closed

- **Local** geometry $\mathcal{O}_{X \cup D}(-D)$: $\dim_{\mathbb{C}} = 4$ toric Calabi-Yau manifold

- **Closed** geometry $\tilde{X}$: semi-projective partial compactification of $\mathcal{O}_{X \cup D}(-D)$

Inclusion $\iota: X \hookrightarrow X \cup D \hookrightarrow \mathcal{O}_{X \cup D}(-D) \hookrightarrow \tilde{X}$

- Curve classes: $\iota_*: H_2(X, L) \longrightarrow H_2(\tilde{X})$
- Insertions: $\iota^*: H^2(\tilde{X}; \mathbb{Q}) \longrightarrow H^2(X; \mathbb{Q})$
Closed Gromov-Witten Invariants

- Virtual counts of stable maps from (borderless) Riemann surfaces to $\tilde{X}$
  - Curve class: $\tilde{\beta} \in H_2(\tilde{X})$
  - Insertions: $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \in H^2(\tilde{X}; \mathbb{Q})$
  - Additional fixed insertion $\tilde{\gamma} \in H^4_{\tilde{T}',(\tilde{X}; \mathbb{Q})}$ supported on fiber over $D$
  - Defined by localization using complex CY 3-subtorus $\tilde{T}'$

$$\langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \tilde{\gamma} \rangle_{\tilde{X},\tilde{\beta}}^f := \int_{[\mathcal{M}_{0,n+1}(\tilde{X},\tilde{\beta})_{\tilde{T}'}^\text{vir}} \left[ \prod_{i=1}^n \text{ev}_i^* \tilde{\gamma}_i \cdot \text{ev}_{n+1}^* \tilde{\gamma} \right] e_{\tilde{T}'}(N^\text{vir}) \mid_{\text{wt restriction}} \in \mathbb{Q}$$
Numerical Open/Closed Correspondence

• Take any $\beta' = \beta + d[B] \in H_2(X, L) \Rightarrow \widetilde{\beta} = \iota_* (\beta') \in H_2(\widetilde{X})$

• $\gamma_1, \ldots, \gamma_n \in H^2(X; \mathbb{Q}) \Rightarrow$ lifts $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n \in H^2(\widetilde{X}; \mathbb{Q})$

Thm (Liu-Y)

We have

$$\langle \gamma_1, \ldots, \gamma_n \rangle^X_{\beta',d} = \langle \widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n, \widetilde{\gamma} \rangle^{\widetilde{X},f}_{\widetilde{\beta}}$$
Numerical Correspondence - Proof by Picture

- Picture gives injective map

\[
\left\{ \text{components of fixed locus of}\right. \\
\left. \text{moduli of open stable maps} \right\} \longrightarrow \left\{ \text{components of fixed locus of}\right. \\
\left. \text{moduli of closed stable maps} \right\}
\]

- Show that additional components on RHS don’t contribute
Numerical Open/Closed Correspondence

Thm (Liu-Y)

We have

\[
\langle \gamma_1, \ldots, \gamma_n \rangle_{\beta',d}^{X,(L,f)} = \langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \tilde{\gamma} \rangle_{\tilde{\beta}}^{\tilde{X},f}
\]

- Both sides are related to relative invariants of \((X \sqcup D, D)\)
- Open/relative: already known [Li-Liu-Liu-Zhou, Fang-Liu]
  - Originates from mathematical theory of topological vertex
  - Holds for all genera and boundary winding profiles
- Relative/closed: instance of log-local principle [van Garrel-Graber-Ruddat]
  - General class of non-compact examples
  - Generalizes examples of [Bousseau-Brini-van Garrel] from Looijenga pairs
Numerical Open/Closed Correspondence

Thm (Liu-Y)

We have

\[ \langle \gamma_1, \ldots, \gamma_n \rangle_{X, (L, f)}^{(L', f')} = \langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n, \tilde{\gamma} \rangle_{\tilde{X}, f}^{\tilde{X}, \tilde{f}} \]

- Potential applications (for future study): structures in open Gromov-Witten theory
  - Open WDVV
  - Open/closed Gopakumar-Vafa invariants
  - ...

Song Yu (Columbia)
• Levels of open/closed correspondence:
  ○ Numerical invariants at individual curve classes ✓
  ○ Generating functions ←
  ○ Givental-style mirror symmetry (J- and I-functions) ←
  ○ B-model mirror families, periods, Picard-Fuchs systems
  ○ Wall-crossings, crepant transformations
Generating Functions

• Setup
  ○ Take basis $u_1, \ldots, u_k \in H^2(X; \mathbb{Q}) \Rightarrow$ lifts $\tilde{u}_1, \ldots, \tilde{u}_k \in H^2(\tilde{X}; \mathbb{Q})$
  ○ Take $\tilde{u}_{k+1} \in H^2(\tilde{X}; \mathbb{Q})$ as class of toric divisor corresponding to $D$
    $\Rightarrow$ completes $\tilde{u}_a$'s into basis
  ○ Set $\tau_2 = \tau_1 u_1 + \cdots + \tau_k u_k$, $\tilde{\tau}_2 = \tilde{\tau}_1 \tilde{u}_1 + \cdots + \tilde{\tau}_{k+1} \tilde{u}_{k+1}$
  ○ $t$: additional variable for open sector

• Generating function of disk invariants:

$$F_{X,L,f}(\tau_2, t) := \sum_{\beta' = \beta + d[B]} \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\langle \tau_2^n \rangle_{X,(L,f)}^{\beta',d}}{n!} t^d$$

• Generating function of closed invariants:

$$\langle \langle \gamma \rangle \rangle_{X,f}(\tilde{\tau}_2) := \sum_{\tilde{\beta}} \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\langle \tilde{\tau}_2^n, \tilde{\gamma} \rangle_{\tilde{X},f}^{\tilde{\beta}}}{n!}$$
Correspondence of Generating Functions

- Generating function of disk invariants:

\[
F_{X,L,f}(\tau_2, t) := \sum_{\beta' = \beta + d[B]} \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\langle \tau_2^n \rangle_{X,(L,f)}^{\beta',d}}{n!} t^d
\]

- Generating function of closed invariants:

\[
\langle \widetilde{\gamma} \rangle \widetilde{X},f(\widetilde{\tau}_2) := \sum_{\beta} \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{\langle \widetilde{\tau}_2^n, \widetilde{\gamma} \rangle_{\widetilde{X},f}^{\beta}}{n!}
\]

**Thm (Liu-Y)**

We have

\[
F_{X,L,f}(\tau_2, t) = \langle \widetilde{\gamma} \rangle \widetilde{X},f(\widetilde{\tau}_2)
\]

under \( \widetilde{\tau}_a = \tau_a \) for \( a = 1, \ldots, k \) and \( \widetilde{\tau}_{k+1} = \log t \).
**Correspondence of Generating Functions**

### Thm (Liu-Y)

We have

\[ F_{X,L,f}(\tau_2, t) = \langle \gamma \rangle^{\bar{X},f}(\bar{\tau}_2) \]

under \( \bar{\tau}_a = \tau_a \) for \( a = 1, \ldots, k \) and \( \bar{\tau}_{k+1} = \log t \).

- From Gromov-Witten theory/quantum cohomology:

  \[ \langle \gamma \rangle^{\bar{X},f} = \left[ z^{-2} \right] \left( 1, S_{\bar{X}}(z) \bar{\gamma} \right)^{\bar{T}'}_{\bar{X}} = \left[ z^{-2} \right] \left( J_{\bar{X}}(z), \bar{\gamma} \right)^{\bar{T}'}_{\bar{X}} \]

  - \( S_{\bar{X}}^{\bar{T}'} \): fundamental solution to \( \bar{T}' \)-equivariant QDE of \( \bar{X} \)
  - \( J_{\bar{X}}^{\bar{T}'} \): \( \bar{T}' \)-equivariant \( J \)-function of \( \bar{X} \)
  - \( \left[ z^{-2} \right] \): taking coefficient of \( z^{-2} \)

### Thm (Liu-Y)

We have

\[ F_{X,L,f}(\tau_2, t) = \left[ z^{-2} \right] \left( J_{\bar{X}}^{\bar{T}'}(\bar{\tau}_2, z), \bar{\gamma} \right)^{\bar{T}'}_{\bar{X}} \bigg|_{\text{wt restriction}} \]
Compatibility with Mirror Symmetry

\[ F^{X,L,f} \longleftrightarrow W^{X,L,f} \]

\[ J_{\tilde{T}' X} \longleftrightarrow I_{\tilde{T}' X} \]

- **Left**: A-model open/closed correspondence

**Thm (Liu-Y)**

We have

\[ F^{X,L,f}(\tau_2, t) = [z^{-2}] \left( J_{\tilde{T}' X}(\tilde{\tau}_2, z), \tilde{\gamma} \right) \bigg|_{\text{wt restriction}} \]
\textbf{Compatibility with Mirror Symmetry}

\[ F^{X,L,f} \leftrightarrow W^{X,L,f} \]

\[ J_{\tilde{X}}^{\tilde{T}'} \leftrightarrow I_{\tilde{X}}^{\tilde{T}'} \]

- **Bottom**: Toric mirror theorem \cite{Givental, Coates-Corti-Iritani-Tseng}

\[ J_{\tilde{X}}^{\tilde{T}'} (\tilde{\tau}_2, z) = I_{\tilde{X}}^{\tilde{T}'} (\tilde{q}, z) \]

under closed mirror map \( \tilde{\tau}_2 = \tilde{\tau}_2(\tilde{q}) \)

- \( I_{\tilde{X}}^{\tilde{T}'} \): explicit hypergeometric function in B-model variables \( \tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_{k+1}) \)
Compatibility with Mirror Symmetry

\[ F^X, L, f \leftrightarrow W^X, L, f \]
\[ J^\tau' \leftrightarrow I^\tau' \]

- **Top:** Open mirror theorem [Fang-Liu, Fang-Liu-Tseng]

\[ F^X, L, f (\tau_2, t) = W^X, L, f (q, x) \]

under closed mirror map \( \tau_2 = \tau_2(q) \) and **open** mirror map \( t = t(q, x) \)

- \( W^X, L, f (q, x) \): explicit hypergeometric function in B-model closed variables \( q = (q_1, \ldots, q_k) \) and open variable \( x \)
Compatibility with Mirror Symmetry

\[ \begin{array}{c}
F^{X,L,f} \leftrightarrow W^{X,L,f} \\
\uparrow \hspace{2cm} \uparrow \\
J_{\tilde{X}}^{T'} \leftrightarrow I_{\tilde{X}}^{T'}
\end{array} \]

- Right: B-model open/closed correspondence

**Thm (Liu-Y)**

We have

\[
W^{X,L,f}(q,x) = [z^{-2}] \left( l_{\tilde{X}}^{T'}(\tilde{q}, z), \tilde{\gamma} \right)_{\tilde{X}}^{T'} |_{\text{wt restriction}}
\]

under \( \tilde{q}_a = q_a \) for \( a = 1, \ldots, k \) and \( \tilde{q}_{k+1} = x \).
Compatibility with Mirror Symmetry

\[
\begin{array}{ccc}
F^{X,L,f} & \leftrightarrow & W^{X,L,f} \\
\uparrow & & \uparrow \\
J_{\widetilde{X}}^{\tilde{T}'} & \leftrightarrow & I_{\widetilde{X}}^{\tilde{T}'}
\end{array}
\]

- Upshot: we establish left/right sides and verify “commutativity” of diagram
- Can recover any of top/left/right from the other two
Current/Future Developments

- Levels of open/closed correspondence:
  - Numerical invariants at individual curve classes ✓
  - Generating functions ✓
  - Givental-style mirror symmetry (J- and I-functions) ✓
  - B-model mirror families, periods, Picard-Fuchs systems ←
  - Wall-crossings, crepant transformations ←
Extended Picard-Fuchs System and Periods

- Initial observation: open mirror map and disk function give solutions to an extension of Picard-Fuchs system of DEs specified by $X$

- And, this extended system coincides with Picard-Fuchs system specified by $\tilde{X}$

- Known: solutions to Picard-Fuchs can be given by periods on Hori-Vafa mirror

$$\int_{\Gamma} \Omega_q, \quad \int_{\tilde{\Gamma}} \tilde{\Omega}_{\tilde{q}}$$

  - $\Gamma \in H_3(X_q^\vee)$, $\tilde{\Gamma} \in H_4(\tilde{X}_{\tilde{q}}^\vee)$
  - $\Omega, \tilde{\Omega}$ holomorphic volume forms

- Work in progress: we find divisor $Y_{q,x} \subset X_q^\vee$ such that

  - Periods over $\Gamma \in H_3(X_q^\vee, Y_{q,x})$ recover open mirror map and disk function
  - $\exists$ isomorphism $\alpha : H_3(X_q^\vee, Y_{q,x}) \sim H_4(\tilde{X}_{\tilde{q}}^\vee)$ such that

$$\int_{\Gamma} \Omega_q = \frac{1}{2\pi \sqrt{-1}} \int_{\alpha(\Gamma)} \tilde{\Omega}_{\tilde{q}}$$
Wall-Crossings and Crepant Transformations

open or closed phase shifts on \((X, L)\) \leftrightarrow \text{closed phase shifts on } \tilde{X}

closed phase shift (crepant resolution) \hspace{1cm} open phase shift (outer to inner)
Thank you!