Mathematics V1202
Calculus IV

Answers to Midterm Exam #1

October 9, 2006
2:40–3:55 pm

1. $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_0^1 \int_0^x e^{x^2}\,dy\,dx = \int_0^1 xe^{x^2}\,dx = \left(\frac{1}{2}e^{x^2}\right)_0^1 = \frac{1}{2}(e-1).$

2. (a) In spherical coordinates, this is

$$\int_0^\pi \int_0^{2\pi} \int_r^R \frac{\rho^2 \sin\phi}{\rho^n}\,d\rho\,d\theta\,d\phi = \left(\int_0^\pi \sin\phi d\phi\right)\left(\int_0^{2\pi} 1 d\theta\right)\left(\int_r^R \rho^{2-n} d\rho\right),$$

which equals $4\pi(R^{3-n} - r^{3-n})/(3-n)$ unless $n = 3$, in which case it's $4\pi(\ln R - \ln r) = 4\pi \ln(R/r)$.

(b) When $n = 3$, we've got the logarithm which doesn't have this limit, but otherwise we've got r^{3-n} which has the limit if and only if $n < 3$.

3. The Jacobian determinant is $\partial u/\partial x\,\partial v/\partial y - \partial u/\partial y\,\partial v/\partial x$, which equals $2x\cos^3 y \sin y + 2x \cos y \sin^3 y = 2x \cos y \sin y = x \sin 2y$. This vanishes precisely when $x = 0$ or y is a multiple of $\pi/2$.

4. Using polar coordinates, the y-moment is

$$\int\!\!\int_C x\,dA = \int_{\pi/4}^{7\pi/4} \int_1^2 (r\cos\theta)\,r\,dr\,d\theta = \left(\int_{\pi/4}^{7\pi/4} \cos\theta\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = -7\sqrt{2}/3.$$

Likewise, the mass is $\int\!\!\int_C 1\,dA = 9\pi/4$ (though you could also work this out from the area πr^2 of a disc). The x-moment is 0 on symmetry grounds, so the center of mass is $(-28\sqrt{2}/27\pi, 0)$.

5. As in the HW problem 49 from §15.7, this is minimized when D is the region where the integrand is ≤ 0. For on any other region E, the integral may be decreased by including the portion of D which is not in E, or by excluding the portion of E which is not in D. The region D is the one enclosed by the ellipse $x^2/4 + y^2 = 1$.

6. Note that $f(x,y) = 2x^2 + 2y^2$ and $g(x,y) = 4xy$ have the same partials! That is, $\partial f/\partial x = \partial g/\partial y = 4x$, and $\partial f/\partial y = \partial g/\partial x = 4y$. The surface area integrand in each case is thus $\sqrt{1 + (4x)^2 + (4y)^2}$. Since the region E enclosed by the ellipse lies inside the rectangle R and the integrand is positive, we have

$$\int\!\!\int_E \sqrt{1 + (4x)^2 + (4y)^2}\,dA < \int\!\!\int_R \sqrt{1 + (4x)^2 + (4y)^2}\,dA,$$

that is, $S < T$.