Mathematics V1202
Calculus IV

Answers to Midterm Exam #2

November 20, 2006
2:40–3:55 pm

1. True by Green's theorem, since $\partial g/\partial x - \partial f/\partial y = 0 - 0 = 0$.

2. False: the integral of $\mathbf{i}$ along the straight line from $\mathbf{0}$ to $\mathbf{j}$ is 0.

3. True: $\mathbf{F}$ has divergence $= 6$, so if E is the region enclosed by S, by the divergence theorem $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 6\ dV = 6 \text{ vol } E \geq 0$.

4. The curl of $\mathbf{F}$ is $-2y\mathbf{i} - 2x\mathbf{j} - \mathbf{k}$. Since $\sin 2t = 2 \sin t \cos t$, C lies on the surface $z = 2xy$. Let S be the part of this surface bounded by C. Then by Stokes's theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$. Note that the parametrization orients C *clockwise* when viewed from above, so if S is oriented *upward* as usual, the sign will have to be reversed. The projection of S onto the x, y-plane is the unit disk D. So parametrize S by $\mathbf{r}(u, v) = (u, v, 2uv)$; then $\mathbf{r}_u \times \mathbf{r}_v = (1, 0, 2v) \times (0, 1, 2u) = (-2v, -2u, 1)$ which orients S upward. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = -\iint_D (-2v, -2u, -1) \cdot (-2v, -2u, 1)\, du\, dv = -\iint_D (4v^2 + 4u^2 - 1)\, dA = -\int_0^{2\pi} \int_0^1 (4r^2 - 1) r\, dr\, d\theta = -2\pi(4r^4/4 - r^2/2)_0^1 = -\pi$.

5. The cross-product $\mathbf{v} \times \nabla f$ equals

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(b\,\frac{\partial f}{\partial z} - c\,\frac{\partial f}{\partial y},\ c\,\frac{\partial f}{\partial x} - a\,\frac{\partial f}{\partial z},\ a\,\frac{\partial f}{\partial y} - b\,\frac{\partial f}{\partial x} \right).$$

Let these be (P, Q, R); then $\nabla \cdot (P, Q, R) = b\frac{\partial^2 f}{\partial x \partial z} - c\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y \partial x} - a\frac{\partial^2 f}{\partial y \partial z} + a\frac{\partial^2 f}{\partial z \partial y} - b\frac{\partial^2 f}{\partial z \partial x}$, which vanishes by Clairaut's theorem on the equality of mixed partials.

6. The fundamental theorem says that if C has endpoints $\mathbf{a}$ and $\mathbf{b}$ (and f has continuous partials), then $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a})$. So the gradient of any function constant on the parabola will work, say $f(x, y) = y - x^2$ and $\mathbf{F} = \nabla f = (-2x, 1)$.

7. The Jacobian determinant is $\frac{\partial(x,y)}{\partial(u,v)} = -13$, the integrand is $x + y = 5u + v$ and by the hint, R is the image of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$. Therefore $\iint_R (x + y)\, dx\, dy = \int_0^1 \int_0^1 (5u + v)|-13|\, du\, dv = 13 \int_0^1 (5/2 + v)\, dv = 13 \cdot 3 = 39$.