Mathematics V1202
Calculus IV

Answers to Midterm Exam #2
November 20, 2006
1:10–2:25 pm

1. False: around a counterclockwise square centered at the origin, the line integral is clearly positive, so the vector field can't be conservative.

2. True: any $\mathbf{v} = (a, b, c)$ is the gradient of $ax + by + cz$ and use the Fundamental Theorem of Line Integrals. Alternative reason: $\nabla \times \mathbf{v} = 0$, so apply Stokes to any surface bounded by C.

3. False: our (least?) favorite vector field $(-y/(x^2 + y^2),\ x/(x^2 + y^2))$ has this property on $U = \mathbf{R}^2 \backslash \{(0, 0)\}$, but its integral around the unit circle is 2π, while its integral on a stationary path is of course 0. Alternative reason: same vector field, but consider the two semicircles from, say, $[-1, 0]$ to $[1, 0]$.

4. Parametrize by $\mathbf{r}(u, v) = (u, v, \sqrt{u^2 + v^2})$ on the annulus A given by $1 \leq u^2 + v^2 \leq 4$.
Then $\mathbf{r}_u = (1, 0, u/\sqrt{u^2 + v^2})$,$\mathbf{r}_v = (0, 1, v/\sqrt{u^2 + v^2})$, and
$\mathbf{r}_u \times \mathbf{r}_v = (-u/\sqrt{u^2 + v^2}, -v/\sqrt{u^2 + v^2}, 1)$, so the integral is

$$\iint_A (-u, -v, u^2 + v^2) \cdot (-u/\sqrt{u^2 + v^2}, -v/\sqrt{u^2 + v^2}, 1)\, du\, dv$$

$$= \int_0^{2\pi} \int_1^2 (r + r^2) r\, dr\, d\theta = 2\pi \left(\frac{r^3}{3} + \frac{r^4}{4} \right)_1^2 = \frac{73\pi}{6}.$$

5. Let $\mathbf{G} = (P, Q, R)$ and abbreviate $\partial/\partial x$ by ∂_x. Then

$$\nabla \times (f\mathbf{G}) = (\partial_y(fR) - \partial_z(fQ), \partial_z(fP) - \partial_x(fR), \partial_x(fQ) - \partial_y(fP)).$$

By the Leibniz rule $\partial_y(fR) = f(\partial_y R) + (\partial_y f)R$ and so on, so the above equals $(f(\partial_y R) + (\partial_y f)R - f(\partial_z Q) - (\partial_z f)Q,\ f(\partial_x P) + (\partial_z f)P - (\partial_x f)R - f(\partial_x R),$ $(\partial_x f)Q + f(\partial_x Q) - (\partial_y f)P - f(\partial_y P))$. On the other hand, the cross product

$$(\nabla f) \times \mathbf{G} = ((\partial_y f)R - (\partial_z f)Q, (\partial_z f)P - (\partial_x f)R, (\partial_x f)Q - (\partial_y f)P)$$

while

$$f(\nabla \times \mathbf{G}) = (f(\partial_y R) - f(\partial_z Q), f(\partial_z P) - f(\partial_x R), f(\partial_x Q) - f(\partial_y P)),$$

which sum to the expression above.

6. $\nabla \cdot \mathbf{F} = y^2 + z^2 + x^2$, so if B is the unit ball, the divergence theorem says

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B (y^2 + z^2 + x^2)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \cdot 2 \cdot \frac{1}{5} = \frac{4\pi}{5}.$$

7. Parametrize C by $\mathbf{r}(t) = (\cos t, \sin t, f(\cos t, \sin t))$ for $t \in [0, 2\pi]$. Then $\mathbf{r}'(t) = (-\sin t, \cos t,$ $-\sin t\, \partial f/\partial x(\cos t, \sin t) + \cos t\, \partial f/\partial y(\cos t, \sin t))$ by the chain rule, so the arclength is

$$\int_0^{2\pi} \sqrt{1 + (-\sin t\, \partial f/\partial x + \cos t\, \partial f/\partial y)^2}\, dt,$$

where the partials are evaluated at $(\cos t, \sin t)$. Since square root is an increasing function and a square is ≥ 0, the integrand is ≥ 1, and hence by the comparison property the integral is $\geq 2\pi$.