Mathematics V1202
Calculus IV

Answers to Final Exam

December 18, 2006
1:10–4 pm

1. The top and bottom of the ellipsoid are described by $z = \pm\sqrt{16 - 4x^2 - 4y^2}$. Since these are always defined when $x^2 + y^2 \le 1$ (that is, the expression under the square root is never negative), E is the region between the graphs of the positive and negative branches, so if D is the unit disk, the volume is $\iint_D 2\sqrt{16 - 4x^2 - 4y^2}\, dA = \int_0^{2\pi} \int_0^1 2\sqrt{16 - 4r^2}\, r\, dr\, d\theta = -\frac{1}{6} 2\pi (16 - 4r^2)^{3/2}|_0^1 = -\frac{\pi}{3}(12^{3/2} - 16^{3/2}) = \frac{8\pi}{3}(8 - 3\sqrt{3})$.

2. Substituting the expressions for u and v into $x^2 - xy + y^2$ and simplifying yields $2u^2 + 2v^2$. So the transformation taking u, v to x, y takes the unit disk D (where $2u^2 + 2v^2 \le 2$) to R. The Jacobian determinant is $\frac{\partial(x,y)}{\partial(u,v)} = \sqrt{2}\sqrt{2/3} + \sqrt{2}\sqrt{2/3} = 4/\sqrt{3}$. Hence by change of variables, $\iint_R (x^2 - xy + y^2)\, dx\, dy = \iint_D (2u^2 + 2v^2)\, \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv = \frac{8}{\sqrt{3}} \iint_D (u^2 + v^2)\, du\, dv = \frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta = \frac{8}{\sqrt{3}} \cdot 2\pi \cdot \frac{1}{4} = \frac{4\pi\sqrt{3}}{3}$.

3. Here $dx = x'(t)dt = 3t^2\, dt$, $dy = y'(t)dt = -2t\, dt$, and $dz = z'(t)dt = dt$, so the line integral $= \int_0^1 (3t^2 \sin(t^3) - 2t\cos(-t^2) + t^4)\, dt = (-\cos(t^3) + \sin(-t^2) + t^5/5)_0^1 = -\cos(1) + \sin(-1) + 6/5$.

4. Here $\mathbf{F}$ is horrendous, but $\nabla \times \mathbf{F} = (0, 0, -3y^2)$, and besides we know C bounds the part S of the hemisphere inside the cylinder. Parametrize S by $\{\mathbf{r}(u,v) = (u, v, \sqrt{16 - u^2 - v^2}) \,|\, (u,v) \in D\}$, where D is the disk $u^2 + v^2 \le 4$. Then (taking $s = \sqrt{16 - u^2 - v^2}$ for brevity) $\mathbf{r}_u = (1, 0, -u/s)$, $\mathbf{r}_v = (0, 1, -v/s)$, and $\mathbf{r}_u \times \mathbf{r}_v = (u/s, v/s, 1)$. Hence by Stokes's theorem $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv = \iint_D (0, 0, -3v^2) \cdot (u/s, v/s, 1)\, du\, dv = \iint_D -3v^2\, du\, dv = \int_0^{2\pi} \int_0^2 3r^2 \sin^2\theta\, r\, dr\, d\theta = 3\int_0^2 r^3\, dr \int_0^{2\pi} \sin^2\theta\, d\theta = 12\pi$.

5. Since both curves are oriented counterclockwise, if R is the plane region between them, $\partial R = D - C$, so by Green's theorem $\iint_R (3x^2 + 3y^2)\, dA = \oint_D \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r}$. But since the integrand is nonnegative, by the comparison property of integrals the left-hand side is ≥ 0. Hence $\oint_D \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r} \ge 0$, or $\oint_D \mathbf{F} \cdot d\mathbf{r} \ge \oint_C \mathbf{F} \cdot d\mathbf{r}$.

6. Let the scalar field be g: then $\partial g/\partial x = e^x yz$ implies $g = e^x yz + C(y,z)$; $\partial g/\partial y = e^x z + e^y z$ implies $\partial C/\partial y = e^y z$ and $C = e^y z + D(z)$; and $\partial g/\partial z = e^x y + e^y + e^z$ implies $\partial D/\partial z = e^z$ and $D = e^z + K$, so finally $g = e^x yz + e^y z + e^z + K$. Then by the FTLI, $\int_C \mathbf{F} \cdot d\mathbf{r} = g(\mathbf{r}(1)) - g(\mathbf{r}(0)) = g(1,1,1) - g(0,0,0) = 3e - 1$.

7. (a) $\nabla \times \mathbf{G} = \nabla \times \nabla f = 0$ since this is a theorem for all scalar fields f, while $\nabla \cdot \nabla f = \nabla \cdot (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0$ since f is harmonic.
(b) A theorem stated that any vector field with zero curl on a simply-connected open set is a gradient. Since $\mathbf{R}^3$ is certainly simply-connected and open, this means $\mathbf{G} = \nabla f$ for some f. And $0 = \nabla \cdot \mathbf{G} = \nabla \cdot \nabla f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$, so f is harmonic.

8. This is $20(1/2 + i\sqrt{3}/2) = 20(\cos\pi/3 + i\sin\pi/3) = 20e^{i\pi/3}$, so its square roots are $2\sqrt{5}e^{i\pi/6}$ and $2\sqrt{5}e^{i7\pi/6}$, or $2\sqrt{5}(\cos\pi/6 + i\sin\pi/6)$ and $2\sqrt{5}(\cos 7\pi/6 + i\sin 7\pi/6)$, or $\pm 2\sqrt{5}(\sqrt{3}/2 + i/2) = \pm(\sqrt{15} + i\sqrt{5})$.

9. Let $u = \ln\sqrt{x^2+y^2}$ and $v = \tan^{-1}(y/x)$. Then by the chain rule twice $\partial u/\partial x = \frac{x/\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}$, while $\partial v/\partial y = \frac{1}{x}\frac{1}{1+(y/x)^2} = \frac{x}{x^2+y^2}$ too. Likewise, $\partial u/\partial y = \frac{y/\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \frac{y}{x^2+y^2}$, while $\partial v/\partial x = \frac{-y}{x^2}\frac{1}{1+(y/x)^2} = \frac{-y}{x^2+y^2}$. So $u+iv$ satisfies the Cauchy-Riemann equations and hence is holomorphic. [It seems like a miracle, but in fact this is the principal value of the complex logarithm…]

10. Since $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$, this is $\int_C x^2+y^2\,(dx + i\,dy) = \int_C x^2+y^2\,dx + i\int_C x^2+y^2\,dy = \int_C (x^2+y^2, 0)\cdot d\mathbf{r} + i\int_C (0, x^2+y^2)\cdot d\mathbf{r}$. Parametrize the line segment from $0 = (0,0)$ to $i = (0,1)$ by $(0,t)$ for $t \in [0,1]$; then $d\mathbf{r} = \mathbf{r}'(t)\,dt = (0,1)\,dt$, so the dot product in the first integral vanishes, and we just get $i\int_0^1 t^2\,dt = i/3$.

11. Let f be a complex function, holomorphic (with continuous partials) on an open set containing a plane region R. Then $\oint_{\partial R} f(z)\,dz = 0$.

12. Note that $\frac{1}{z^2-1} = \frac{1}{z-1}\frac{1}{z+1}$, and $f(z) = \frac{1}{z+1}$ is holomorphic on $\mathbf{C}\setminus\{-1\}$, an open set containing the region enclosed by C (which is a disk of radius $\sqrt{2}$ centered at 1). Hence by the Cauchy Integral Formula, the integral is $2\pi i f(1) = \pi i$.