Mathematics V1202
Calculus IV

Answers to Final Exam
December 20, 2006
1:10–4 pm

1. Let $f(x,y)$ be a function continuous on $R=[a,b]\times[c,d]$. Then $\int_a^b\int_c^d f(x,y)\,dy\,dx = \iint_R f\,dA$ (which therefore also equals $\int_c^d\int_a^b f(x,y)\,dx\,dy$). More generally, this is true if f is bounded and discontinuous only on a finite number of piecewise smooth parametric curves and both iterated integrals exist.

2. By rotational symmetry the center of mass will be on the z-axis, so we only have to do two integrals, and we can assume the density is 1. Then the mass $m=\iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}1\,d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^1\rho^2\,d\rho = 2\pi\cdot\frac{2-\sqrt{2}}{2}\cdot\frac{1}{3} = \frac{\pi(2-\sqrt{2})}{3}$. And since $z=\rho\cos\phi$, the moment $M_{xy}=\iiint_E z\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$ $=\int_0^{2\pi}1\,d\theta\int_0^{\pi/4}\frac{1}{2}\sin(2\phi)\,d\phi\int_0^1\rho^3\,d\rho = 2\pi\cdot-\frac{1}{4}\cos(2\phi)|_0^{\pi/4}\cdot\frac{1}{4}=\pi/8$. Hence the center of mass is $(0,0,\frac{3}{8(2-\sqrt{2})}) = (0,0,\frac{3(2+\sqrt{2})}{16})$.

3. Indeed, $\mathbf{F}$ is conservative: indeed, $\mathbf{F}=\nabla g$ for $g(x,y,z)=e^{xz}$ (you can derive the antiderivative by the methods we studied, or just guess it). By the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F}\cdot d\mathbf{r} = g(\mathbf{r}(2\pi)) - g(\mathbf{r}(0)) = g(1,0,2\pi)-g(1,0,0) = e^{2\pi}-1$.

4. This is a surface integral. Parametrize the net by $\mathbf{r}(u,v) = u\cos v, u\sin v, u)$ for $(u,v)\in[0,2]\times[0,2\pi]$. Then $\mathbf{r}_u=(\cos v,\sin v,1)$, $\mathbf{r}_v=(-u\sin v,u\cos v,0)$, and $\mathbf{r}_u\times\mathbf{r}_v = (-u\cos v,-u\sin v,u)$ which points upward. The surface integral then equals $\int_0^2\int_0^{2\pi}\mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,du\,dv$
$=\int_0^2\int_0^{2\pi}(u\sin v,-u\cos v,u^2\sin^2 v)\cdot(-u\cos v,-u\sin v,u)\,du\,dv$
$=\int_0^2\int_0^{2\pi}u^3\sin^2 v\,du\,dv = \int_0^2 u^3\,du\int_0^{2\pi}\sin^2 v\,dv = 4\pi$.

5. (a) This is the portion of the unit sphere centered at 0 between $z=a$ and $z=b$. (b) Here (with the abbreviation $s=\sqrt{1-u^2}$) $\mathbf{r}_u = (-u\cos v/s, -u\sin v/s, 1)$, $\mathbf{r}_v=(-s\sin v, s\cos v, 0)$. and $\mathbf{r}_u\times\mathbf{r}_v = (-s\cos v,-s\sin v,-u)$, so the surface area is $\int_a^b\int_0^{2\pi}|\mathbf{r}_u\times\mathbf{r}_v|\,vd\,du = \int_a^b\int_0^{2\pi}\sqrt{s^2+u^2}\,vd\,du = \int_a^b\int_0^{2\pi}1\,dv\,du = 2\pi(b-a)$. [Observe how amazing this is: the surface area of a slice of a sphere only depends on the width of the slice...]

6. (a) If $\mathbf{F}=(P,Q,R)$, then $P=x/(x^2+y^2+z^2)^{3/2}$, and

$$\frac{\partial P}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2}-3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3};$$

there are similar expressions for $\partial Q/\partial y$ and $\partial R/\partial z$ with $3x^2$ replaced by $3y^2$ and $3z^2$ respectively, so the sum of all three is $\nabla \cdot \mathbf{F} = 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} = 0$. (b) Do the surface integral of $\mathbf{F}$ on the unit sphere S: there, $\mathbf{F}(\mathbf{r}) = \mathbf{r}$, and the outward unit normal is also $\mathbf{r}$, so the surface integral is the integral of 1, which is the surface area of the sphere, namely 4π. But if it were true that $F = \nabla \times \mathbf{G}$, by Stokes's theorem $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{G} \cdot d\mathbf{S} = \oint_{\partial \mathbf{S}} \mathbf{G} \cdot d\mathbf{r} = 0$ since ∂S is empty.

7. The divergence is $\nabla \cdot \mathbf{F} = 2x + 2y + 2z$, so by the divergence theorem this equals $2\iiint_E x+y+z\, dV = 2\iint_D \int_0^{x+2} x+y+z \; dz\, dx\, dy = 2\iint_D (xz+yz+z^2/2)|_{z=0}^{z=3}\, dx\, dy = 2\iint_D (3x+3y+9/2)\, dx\, dy = 2\cdot 9/2\cdot\pi = 9\pi$. Here in the penultimate equality we used $\iint_D x\, dx\, dy = \iint_D y\, dx\, dy = 0$ by symmetry and $\iint_D 1\, dx\, dy = \pi$ since this is the area of the unit disk.

8. $(1+i) = \sqrt{2}(1/\sqrt{2} + i/\sqrt{2}) = \sqrt{2}(\cos\pi/4 + i\sin\pi/4) = \sqrt{2}e^{i\pi/4}$, so $(1+i)^{-18} = (\sqrt{2}e^{i\pi/4})^{-18} = \sqrt{2}^{-18}e^{-18i\pi/4} = \frac{1}{2^9}e^{-4i\pi}e^{-i\pi/2} = \frac{1}{512}\cdot 1 \cdot -1 = -i/512$.

9. If $z = x+iy$, then $z + 1/z = z + \bar{z}/z\bar{u}z = x + iy + \frac{x-iy}{x^2+y^2} = (x + \frac{x}{x^2+y^2}) + i(y - \frac{y}{x^2+y^2})$. (a) This is real if and only if its imaginary part vanishes, that is, $y = \frac{y}{x^2+y^2}$, that is, either $y = 0$ or it can be cancelled and $x^2+y^2 = 1$. So either z is real or $|z| = 1$; the sketch consists of the x-axis and the unit circle. (b) Likewise, it's imaginary if and only if its real part vanishes, that is, $x = \frac{-x}{x^2+y^2}$, that is, either $x = 0$ or $x^2+y^2 = -1$. But the latter is impossible since x and y are real, so all we have is $x = 0$, that is, z is imaginary; the sketch consists of the y-axis only.

10. On $C = \{e^{it}\,|\,t \in [0,\pi]$ we have $dz = ie^{it}\,dt$ and $\bar{z} = e^{-it}$, so $\int_C \bar{z}^2\, dz = \int_0^\pi e^{-2it}ie^{it}\, dt = i\int_0^\pi e^{-it}\, dt = i(ie^{-it})_0^\pi = -(-1-1) = 2$.

11. No, $f(x+iy) = (x-iy)^2 = (x^2-y^2) + i(-2xy)$, so $\partial u/\partial x = 2x$ but $\partial v/\partial y = -2x$, which violates the Cauchy-Riemann equations.

12. Expand this out to $\oint_C 2z+5+2z^{-1}\, dz$. The first two terms are holomorphic everywhere, so their integrals around a closed curve vanish by the Cauchy Integral Theorem. The last term is of the form $f(z)/(z-z_0)$ for $f(z) = 2$ holomorphic and $z_0 = 0$ enclosed by the ellipse, so by the Cauchy Integral Formula its integral is $2\pi i f(0) = 4\pi i$.