Mathematics V1202
Calculus IV

Final Examination
December 18, 2006
1:10–4 pm

1. Use polar coordinates to find the volume of the space region E inside both the cylinder $x^2 + y^2 = 1$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 16$.

2. Use the change of variables $x = \sqrt{2}\,u - \sqrt{2/3}\,v$ and $y = \sqrt{2}\,u + \sqrt{2/3}\,v$ to evaluate $\iint_R (x^2 - xy + y^2)\,dx\,dy$, where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

3. Evaluate the line integral $\int_C \sin x\,dx + \cos y\,dy + xz\,dz$ where $C = \{(t^3, -t^2, t) \mid t \in [0,1]\}$.

4. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = (y^3, 1, z + e^{z^7})$ and C is the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, oriented counterclockwise.

5. Let $\mathbf{F}(x,y) = (-y^3, x^3)$. Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} \leq \oint_D \mathbf{F} \cdot d\mathbf{r}$ when C and D are the curves shown in the figure below. Give a clear *reason* at each step.

6. Find a scalar field whose gradient is $\mathbf{F}(x,y,z) = (e^x yz, e^x z + e^y z, e^x y + e^y + e^z)$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C = \{(t, t^2, t^3) \mid t \in [0,1]\}$.

7. (a) Let f be a scalar field (with continuous partials) defined on all of $\mathbf{R}^3$ which is *harmonic*, that is, $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0$. Show that $\mathbf{G} = \nabla f$ has zero curl and zero divergence.

(b) Let $\mathbf{G}$ be a vector field (with continuous partials) defined on all of $\mathbf{R}^3$ with zero curl and zero divergence. Show that $\mathbf{G} = \nabla f$ for some harmonic f.

8. Express the square roots of the complex number $10 + 10\sqrt{3}i$ in rectangular form.

9. Consider the function $\ln \sqrt{x^2 + y^2} + i \tan^{-1}(y/x)$ defined on $\{x + iy \in \mathbf{C} \mid x \neq 0\}$. Is it holomorphic? Why or why not? Hint: $\frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2}$.

10. Evaluate the complex line integral $\int_C z\bar{z}\,dz$, where C is the line segment from 0 to i.

11. Clearly and completely state the Cauchy Integral Theorem (not to be confused with the Cauchy Integral Formula).

12. Evaluate the complex line integral

$$\oint_C \frac{dz}{z^2 - 1},$$

where C is the circle $(x-1)^2 + y^2 = 2$, oriented counterclockwise.