Mathematics V1202
Calculus IV

Final Examination
December 20, 2006
1:10–4 pm

1. Clearly, completely, concisely, and correctly state Fubini's Theorem.

2. Find the center of mass of the ice-cream-cone-shaped region (with constant density) where $x^2+y^2+z^2 \le 1$, $x^2+y^2 \le z^2$, and $z \ge 0$.

3. Let C be a single turn of a helix: $C = \{(\cos t, \sin t, t) \mid t \in [0, 2\pi]\}$. If $\mathbf{F}(x,y,z) = (ze^{xz}, 0, xe^{xz})$, compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. Hint: is $\mathbf{F}$ conservative?

4. A net is cast in the sea in the shape of a cone, $x^2+y^2 = z^2 \le 4$, $z \ge 0$. If the water at (x,y,z) flows along $(y, -x, y^2)$, what is the total flux of water upward through the net? (Don't worry about units.)

5. (a) For $-1 \le a < b \le 1$, name and sketch the surface given in parametric form as

$$S = \{(\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u) \mid (u,v) \in [a,b] \times [0, 2\pi]\}.$$

(b) Compute its surface area.

6. (a) Show that the vector field $\mathbf{F}(\mathbf{r}) = \mathbf{r}/|\mathbf{r}|^3$, defined on $\mathbf{R}^3 \setminus \{0\}$, has zero divergence.

(b) Use Stokes's theorem to show, however, that $\mathbf{F}$ is not the curl of any other vector field $\mathbf{G}$ defined on $\mathbf{R}^3 \setminus \{0\}$.

7. Let $\mathbf{F}(x,y,z) = (x^2+ye^z, y^2+ze^x, z^2+xe^y)$, and let S be the boundary surface of the solid $E = \{(x,y,z) \mid x^2+y^2 \le 1,\ 0 \le z \le 3\}$. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

8. Express $(1+i)^{-18}$ in the form $a+bi$.

9. (a) Describe and sketch all complex numbers z such that $z + 1/z$ is real.

(b) Describe and sketch all complex numbers z such that $z + 1/z$ is imaginary.

10. Evaluate the complex line integral $\int_C \bar{z}^2\, dz$, where C is the upper unit semicircle $x^2+y^2=1$, $y \ge 0$, oriented to the left.

11. Is $f(z) = \bar{z}^2$ a holomorphic function? Why or why not?

12. Evaluate the complex line integral $\oint_C (1+2z)(1+2z^{-1})\, dz$, where C is the ellipse $x^2+9y^2=9$ oriented counterclockwise. State clearly what theorems you are using.