2 Complex Functions and the Cauchy-Riemann Equations

2.1 Complex functions

In one-variable calculus, we study functions $f(x)$ of a real variable x. Likewise, in complex analysis, we study functions $f(z)$ of a complex variable $z \in \mathbb{C}$ (or in some region of $\mathbb{C}$). Here we expect that $f(z)$ will in general take values in $\mathbb{C}$ as well. However, it will turn out that some functions are better than others. Basic examples of functions $f(z)$ that we have already seen are: $f(z) = c$, where c is a constant (allowed to be complex), $f(z) = z$, $f(z) = \bar{z}$, $f(z) = \operatorname{Re} z$, $f(z) = \operatorname{Im} z$, $f(z) = |z|$, $f(z) = e^z$. The "functions" $f(z) = \arg z$, $f(z) = \sqrt{z}$, and $f(z) = \log z$ are also quite interesting, but they are not well-defined (single-valued, in the terminology of complex analysis).

What is a complex valued function of a complex variable? If $z = x + iy$, then a function $f(z)$ is simply a function $F(x, y) = u(x, y) + iv(x, y)$ of the two real variables x and y. As such, it is a function (mapping) from $\mathbb{R}^2$ to $\mathbb{R}^2$. For example, $f(z) = z$ corresponds to $F(x, y) = x + iy$; $f(z) = \bar{z}$ to $F(x, y) = x - iy$, $f(z) = |z|$ to $F(x, y) = \sqrt{x^2 + y^2}$. Here, this last example takes values just along the real axis. If $f(z) = u + iv$, then the function $u(x, y)$ is called the *real part* of f and $v(x, y)$ is called the *imaginary part* of f. Of course, it will not in general be possible to plot the graph of $f(z)$, which will lie in $\mathbb{C}^2$, the set of ordered pairs of complex numbers, but it is the set $\{(z, w) \in \mathbb{C}^2 : w = f(z)\}$. The graph can also be viewed as the subset of $\mathbb{R}^4$ given by $\{(x, y, s, t) : s = u(x, y), t = v(x, y)\}$. In particular, it lies in a four-dimensional space.

The usual operations on complex numbers extend to complex functions: given a complex function $f(z) = u + iv$, we can define functions $\operatorname{Re} f(z) = u$, $\operatorname{Im} f(z) = v$, $\overline{f(z)} = u - iv$, $|f(z)| = \sqrt{u^2 + v^2}$. Likewise, if $g(z)$ is another complex function, we can define $f(z)g(z)$ and $f(z)/g(z)$ for those z for which $g(z) \neq 0$.

Some of the most interesting examples come by using the algebraic operations of $\mathbb{C}$. For example, a *polynomial* is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where the a_i are complex numbers, and it defines a function in the usual way. It is easy to see that the real and imaginary parts of a polynomial $P(z)$ are polynomials in x and y. For example,

$$P(z) = z^2 = x^2 - y^2 + 2xyi.$$

But given two (real) polynomial functions $u(x, y)$ and $z(x, y)$, it is very rarely the case that there exists a complex polynomial $P(z)$ such that $P(z) = u+iv$. For example, it is not hard to see that x cannot be of the form $P(z)$, nor can $\bar{z}$. As we shall see later, no polynomial in x and y taking only real values for every z (i.e. $v = 0$) can be of the form $P(z)$. Of course, since $x = \dfrac{1}{2}(z + \bar{z})$ and $y = \dfrac{1}{2i}(z - \bar{z})$, every polynomial in x and y is also a polynomial in z and $\bar{z}$. Finally, while on the subject of polynomials, let us mention the

Fundamental Theorem of Algebra (first proved by Gauss in 1799): If $P(z)$ is a nonconstant polynomial, then $P(z)$ has a complex root. In other words, there exists a complex number c such that $P(c) = 0$. From this, it is easy to deduce the following corollaries:

1. If $P(z)$ is a polynomial of degree $n > 0$, then $P(z)$ can be factored into linear factors:

$$P(z) = a(z - c_1) \cdots (z - c_n),$$

for complex numbers a and $c_1, \dots, c_n$.

2. Every nonconstant polynomial $p(x)$ with real coefficients can be factored into (real) polynomials of degree one or two.

Here the first statement is a consequence of the fact that c is a root of $P(z)$ if and only if $(z-c)$ divides $P(z)$, plus induction. The second statement follows from the first and the fact that, for a polynomial with **real** coefficients, complex roots occur in conjugate pairs.

One consequence of the Fundamental Theorem of Algebra is that, having enlarged the real numbers so as to have a root of the polynomial equation $x^2 + 1 = 0$, we are now miraculously able to find roots of **every** polynomial equation, including the ones where the coefficients are allowed to be complex. This suggests that it is very hard to further enlarge the complex numbers in such a way as to have any reasonable algebraic properties. Finally, we should mention that, despite its name, the Fundamental Theorem of Algebra is not really a theorem in algebra, and in fact some of the most natural proofs of this theorem are by using methods of complex analysis.

We can define a broader class of complex functions by dividing polynomials. By definition, a *rational function* $R(z)$ is a quotient of two polynomials:

$$R(z) = P(z)/Q(z),$$

where $P(z)$ and $Q(z)$ are polynomials and $Q(z)$ is not identically zero. Using the factorization (1) above, it is not hard to see that, if $R(z)$ is not actually a polynomial, then it fails to be defined, roughly speaking, at the roots of $Q(z)$ which are not also roots of $P(z)$. (We have to be a little careful if there are multiple roots.)

Finally, there are complex functions which can be defined by power series. We have already seen the most important example of such a function, $e^z = \sum_{n=0}^{\infty} z^n/n!$, which is defined for all z. Other examples are, for instance,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad |z| < 1.$$

However, to make sense of such expressions, we would have to discuss convergence of sequences and series for complex numbers. We will not do so here, but will give a brief discussion below of limits and continuity for complex functions. (It turns out that, once things are set up correctly, the comparison and ration tests work for complex power series.)

2.2 Limits and continuity

The absolute value measures the distance between two complex numbers. Thus, z_1 and z_2 are close when $|z_1 - z_2|$ is small. We can then define the *limit* of a complex function $f(z)$ as follows: we write

$$\lim_{z \to c} f(z) = L,$$

where c and L are understood to be complex numbers, if the distance from $f(z)$ to L, $|f(z) - L|$, is small whenever $|z - c|$ is small. More precisely, if we want $|f(z) - L|$ to be less than some small specified positive real nmber ϵ, then there should exist a positive real number δ such that, if $|z - c| < \delta$, then $|f(z) - L| < \epsilon$. Note that, as with real functions, it does not matter if $f(c) = L$ or even that $f(z)$ be defined at c. It is easy to see that, if $c = (c_1, c_2)$, $L = a + bi$ and $f(z) = u + iv$ is written as a real and an part, then $\lim_{z \to c} f(z) = L$ if and only if $\lim_{(x,y) \to (c_1,c_2)} u(x,y) = a$ and $\lim_{(x,y) \to (c_1,c_2)} v(x,y) = b$. Thus the story for limits of functions of a complex variable is the same as the story for limits of real valued functions of the variables x, y. However, a real variable x can approach a real number c only from above or below, whereas there are many ways for a complex variable to approach a complex number c.

Sequences, limits of sequences, and series can be defined similarly.

As for functions of a real variable, a function $f(z)$ is *continuous* at c if

$$\lim_{z \to c} f(z) = f(c).$$

In other words: 1) the limit exists; 2) $f(z)$ is defined at c; 3) its value at c is the limiting value. A function $f(z)$ is *continuous* if it is continuous at al points where it is defined. It is easy to see that a function $f(z) = u + iv$ is continuous if and only if its real and imaginary parts are continuous, and that the usual functions $z, \bar{z}, \operatorname{Re} z, \operatorname{Im} z, |z|, e^z$ are continuous. All polynomials $P(z)$ are continuous, as are all two-variable polynomial functions in x and y. A rational function $R(z) = P(z)/Q(z)$ is continuous where it is defined, i.e. where the denominator is not zero. More generally, if $f(z)$ and $g(z)$ are continuous, then so are:

1. $cf(z)$, where c is a constant;
2. $f(z) + g(z)$;
3. $f(z) \cdot g(z)$;
4. $f(z)/g(z)$, where defined (i.e. where $g(z) \neq 0$).

2.3 Complex derivatives

Having discussed some of the basic properties of functions, we ask now what it means for a function to have a *complex* derivative. Here we will see something quite new: this is very different from asking that its real and imaginary parts have partial derivatives with respect to x and y. We will not worry about the meaning of the derivative in terms of slope, but only ask that the usual difference quotient exists.

Definition A function $f(z)$ is *complex differentiable* at c if

$$\lim_{z \to c} \frac{f(z) - f(c)}{z - c}$$

exists. In this case, the limit is denoted by $f'(c)$. Making the change of variable $z = c + h$, $f(z)$ is complex differentiable at c if and only if the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists, in which case the limit is again $f'(c)$. A function is simply complex differentiable if it is complex differentiable at every point where it is defined, and we write the derivative as the function $f'(z)$ or $\dfrac{d}{dz} f(z)$.

For example, a constant function $f(z) = C$ is everywhere complex differentiable and its derivative $f'(z) = 0$. The function $f(z) = z$ is also complex differentiable, since in this case

$$\frac{f(z) - f(c)}{z - c} = \frac{z - c}{z - c} = 1.$$

Thus $(z)' = 1$. But many simple functions do not have complex derivatives. For example, consider $f(z) = \operatorname{Re} z = x$. We show that the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

does not exist for any c. Let $c = a + bi$, so that $f(c) = a$. First consider $h = t$ a real number. Then $f(c + t) = a + t$ and so

$$\frac{f(c + h) - f(c)}{h} = \frac{a + t - a}{t} = 1.$$

So if the limit exists, it must be 1. On the other hand, we could use $h = it$. In this case, $f(c + it) = f(c) = a$, and

$$\frac{f(c + h) - f(c)}{h} = \frac{a - a}{t} = 0.$$

Thus approaching c along horizontal and vertical directions has given two different answers, and so the limit cannot exist. Other simple functions which can be shown not to have complex derivatives are $\operatorname{Im} z, \bar{z}$, and $|z|$.

On the bright side, the usual rules for derivatives can be checked to hold:

1. If $f(z)$ is complex differentiable, then so is $cf(z)$, where c is a constant, and $(cf(z))' = cf'(z)$;

2. (Sum rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z) + g(z)$, and $(f(z) + g(z))' = f'(z) + g'(z)$;

3. (Product rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z) \cdot g(z)$ and $(f(z) \cdot g(z))' = f'(z)g(z) + f(z)g'(z)$;

4. (Quotient rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(z)/g(z)$, where defined (i.e. where $g(z) \neq 0$), and

$$\left(\frac{f(z)}{g(z)}\right)' = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2};$$

5. (Chain rule) If $f(z)$ and $g(z)$ are complex differentiable, then so is $f(g(z))$ where defined, and $(f(g(z)))' = f'(g(z)) \cdot g'(z)$.

6. (Inverse functions) If $f(z)$ is complex differentiable and one-to-one, with nonzero derivative, then the inverse function $f^{-1}(z)$ is also differentiable, and
$$(f^{-1}(z))' = 1/f'(f^{-1}(z)).$$

Thus for example we have the power rule $(z^n)' = nz^{n-1}$, every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is complex differentiable, with

$$P'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} \cdots + a_1,$$

and every rational function is also complex differentiable. It follows that a function which is not complex differentiable, such as $\operatorname{Re} z$ or $\bar{z}$ cannot be written as a complex polynomial or rational function.

2.4 The Cauchy-Riemann equations

We now turn systematically to the question of deciding when a complex function $f(z) = u + iv$ is complex differentiable. If the complex derivative $f'(z)$ is to exist, then we should be able to compute it by approaching z along either horizontal or vertical lines. Thus we must have

$$f'(z) = \lim_{t \to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t \to 0} \frac{f(z+it) - f(z)}{it},$$

where t is a real number. In terms of u and v,

$$\lim_{t \to 0} \frac{f(z+t) - f(z)}{t} = \lim_{t \to 0} \frac{u(x+t,y) + iv(x+t,y) - u(x,y) - v(x,y)}{t}$$
$$= \lim_{t \to 0} \frac{u(x+t,y) - u(x,y)}{t} + i \lim_{t \to 0} \frac{v(x+t,y) - v(x,y)}{t} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Taking the derivative along a vertical line gives

$$\lim_{t \to 0} \frac{f(z+it) - f(z)}{it} = -i \lim_{t \to 0} \frac{u(x,y+t) + iv(x,y+t) - u(x,y) - v(x,y)}{t}$$
$$= -i \lim_{t \to 0} \frac{u(x,y+t) - u(x,y)}{t} + \lim_{t \to 0} \frac{v(x,y+t) - v(x,y)}{t} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Equating real and imaginary parts, we see that: *If a function $f(z) = u + iv$ is complex differentiable, then its real and imaginary parts satisfy the*

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Moreover, the complex derivative $f'(z)$ is then given by

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.}$$

Examples: the function $z^2 = (x^2 - y^2) + 2xyi$ satisfies the Cauchy-Riemann equations, since

$$\frac{\partial}{\partial x}(x^2 - y^2) = 2x = \frac{\partial}{\partial y}(2xy) \quad \text{and} \quad \frac{\partial}{\partial x}(2xy) = 2y = -\frac{\partial}{\partial y}(x^2 - y^2).$$

Likewise, $e^z = e^x \cos y + ie^x \sin y$ satisfies the Cauchy-Riemann equations, since

$$\frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y = \frac{\partial}{\partial y}(e^x \sin y) \text{ and } \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y = -\frac{\partial}{\partial y}(e^x \cos y).$$

Moreover, e^z is in fact complex differentiable, and its complex derivative is

$$\frac{d}{dz}e^z = \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial x}(e^x \sin y) = e^x \cos y + e^x \sin y = e^z.$$

The chain rule then implies that $\frac{d}{dz}e^{\alpha z} = \alpha e^{\alpha z}$. From the sum rule and the expressions for $\sin z$ and $\cos z$ in terms of e^{iz} and e^{-iz}, it is easy to check that the usual rules hold:

$$\frac{d}{dz}\cos z = -\sin z; \qquad \frac{d}{dz}\sin z = \cos z.$$

On the other hand, $\bar{z}$ does not satisfy the Cauchy-Riemann equations, since

$$\frac{\partial}{\partial x}(x) = 1 \neq \frac{\partial}{\partial y}(-y).$$

Likewise, $f(z) = x^2 + iy^2$ does not. Note that the Cauchy-Riemann equations are **two** equations for the partial derivatives of u and v.

We have seen that a function with a complex derivative satisfies the Cauchy-Riemann equations. In fact, the converse is true:

Theorem: Let $f(z) = u + iv$ be a complex function defined in a region (open subset) D of $\mathbb{C}$, and suppose that u and v have continuous first partial derivatives with respect to x and y. If u and v satisfy the Cauchy-Riemann equations, then $f(z)$ has a complex derivative.

The proof of this theorem is not difficult, but involves a more careful understanding of the meaning of the partial derivatives and linear approximation in two variables.

Thus we see that the Cauchy-Riemann equations give a complete criterion for deciding if a function has a complex derivative. There is also a geometric interpretation of the Cauchy-Riemann equations. Recall that $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$ and that $\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$. Then u and v satisfy the Cauchy-Riemann equations if and only if

$$\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right).$$

If this holds, then the level curves $u = c_1$ and $v = c_2$ are orthogonal where they intersect (and the converse is almost true).

Instead of saying that a function $f(z)$ has a complex derivative, or equivalently satisfies the Cauchy-Riemann equations, we shall call $f(z)$ *analytic*. Here are some basic properties of analytic functions, which are easy consequences of the Cauchy-Riemann equations:

Theorem: Let $f(z)$ be an analytic function.

1. If $f'(z)$ is identically zero, then $f(z)$ is a constant.
2. If either $\operatorname{Re} f(z) = u$ or $\operatorname{Im} f(z) = v$ is constant, then $f(z)$ is constant. In particular, a nonconstant analytic function cannot take only real or only pure imaginary values.
3. If $|f(z)|$ is constant or $\arg f(z)$ is constant, then $f(z)$ is constant.

For example, if $f'(z) = 0$, then

$$0 = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$. By the Cauchy-Riemann equations, $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0$ as well. Hence $f(z)$ is a constant. This proves (1). To see (2), assume for instance that u is constant. Then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$, and, as above, the

Cauchy-Riemann equations then imply that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0$. Again, $f(z)$ is constant. Part (3) can be proved along similar but more complicated lines.

2.5 Harmonic functions

Let $f(z)$ be an analytic function, and assume that u and v have partial derivatives of order 2 (in fact, this turns out to be automatic. Then, using the Cauchy-Riemann equations and the equality of mixed partials, we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}.$$

In other words, u satisfies:

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.}$$

The above equation is a very important second order partial differential equation, and solutions of it are called *harmonic functions*. Thus, the real part of an analytic function is harmonic. A similar argument shows that v is also harmonic, i.e. the imaginary part of an analytic function is harmonic. It can also be shown that, essentially, all harmonic functions arise as the real parts of analytic functions. The only slight problem is that the analytic functions might not be single-valued, even if the harmonic function is single valued. The basic example is $\operatorname{Re}\log z = \frac{1}{2}\ln(x^2 + y^2)$. A calculation (left as homework) shows that this function is harmonic. But an analytic function whose real part is the same as that of $\log z$ must agree with $\log z$ up to an imaginary constant, and so cannot be single-valued.

Thus, we can generate lots of harmonic functions, in fact essentially all of them, by taking real or imaginary parts of analytic functions. Harmonic functions are very important in mathematical physics, and one reason for the importance of analytic functions is their connection to harmonic functions.

2.6 Homework

1. Write each of the following functiosn $f(z)$ in the form $u + iv$. Which functions are analytic?

   (a) $z + iz^2$; (b) $1/z$; (c) $\bar{z}/z$.

2. If $f(z) = e^z$, describe the images under $f(z)$ of horizontal and vertical lines, i.e. what are the sets $f(a+it)$ and $f(t+ib)$, where a and b are constants and t runs through all real numbers?

3. Can the function $\bar{z}/z$ be continuously extended to $z = 0$? Why or why not? Is the function $\bar{z}/z$ analytic where it is defined? Why or why not?

4. Let $f(z)$ be a complex function. Is it possible for both $f(z)$ and $\overline{f(z)}$ to be analytic?

5. Let $f(z) = u + iv$ be analytic. Recall that the *Jacobian* is the function given by the following determinant:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}.$$

   Using the Cauchy-Riemann equations, show that this is the same as $|f'(z)|^2$.

6. Verify that $\operatorname{Re} 1/z$, $\operatorname{Im} 1/z$, and $\operatorname{Re} \log z = \frac{1}{2}\ln(x^2+y^2)$ are harmonic.

7. If $f(z) = u + iv$ is a complex function such that u and v are both harmonic, is $f(z)$ necessarily analytic?