$\mathbb{C}$ is the complex plain

Calc 11/22/06

New Product in $\mathbb{R}^2$:

$(a,b)(c,d) = (ac - bd, bc + ad)$

Write $1 = (1,0)$, $i = (0,1)$

Then $i^2 = (0,1)(0,1) = (-1,0) = -1 \cdot 1 = -1$

and $(a,b) = a(1,0) + b(0,1) = a + bi$

where $(a+bi)(c+di) = (ac-bd) + (bc+ad)i$

let $\mathbb{C} = \mathbb{R}^2 = \{a+bi \mid a, b \in \mathbb{R}\}$

a is the real part and b is the imaginary part

note: $a + bi = c + di$

$\Leftrightarrow$ $a = c$ and $b = d$

why this operation?

Forced by $i^2 = -1$ and distributivity

Also, can divide by any nonzero complex # as follows:

If $z = a + bi$, what is $\frac{1}{z}$?

$$\frac{1}{z} = \frac{1}{a+bi}\,\frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Let $\bar{z} = a - bi$ be called the conjugate of z;

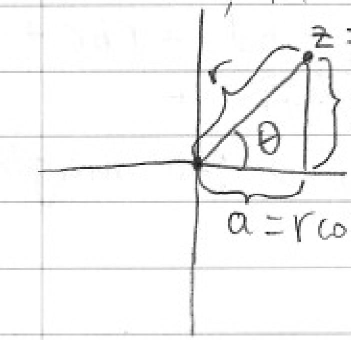
let $|z| = \sqrt{a^2+b^2}$ be the magnitude or modulus of z.

then $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ and $\frac{w}{z} = \frac{w\bar{z}}{|z|^2}$

E.g.

$$\frac{5+2i}{6+i} = \frac{5+2i}{6+i}\,\frac{6-i}{6-i} = \frac{32+7i}{37} = \frac{32}{37} + \frac{7}{37}i$$

note: $|z| = 0 \iff z = 0$,
so can divide by any nonzero complex z.
Indeed, if $r = |z|$



polar form.

$z = r(\cos\theta + i\sin\theta)$

$a = r\cos\theta$

$b = r\sin\theta$

$r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right) \pmod{\pi}$

$x \equiv y \pmod{z}$
means $x - y = nz$
for some integer n
E.g. $22 \equiv 8 \pmod{7}$
because $22 - 8 = 2 \cdot 7$

If $z = r(\cos\theta + i\sin\theta)$ and $w = s(\cos\phi + i\sin\phi)$,
then $wz = s(\cos\phi + i\sin\phi)\,r(\cos\theta + i\sin\theta)$

$= sr((\cos\phi\cos\theta - \sin\phi\sin\theta) + i(\cos\phi\sin\theta + \sin\phi\cos\theta))$

$= sr(\cos(\phi+\theta) + i\sin(\phi+\theta))$

In other words when 2 complex #
are multiplied, the moduli multiply
and the arguments add

$$\underset{\substack{\uparrow\\ \text{modulus}}}{w = s}(\cos\underset{\substack{\uparrow\\ \text{argument}}}{\phi} + i\sin\phi)$$

likewise, taking $r=1$, $z=w=\cos\theta+i\sin\theta$

$$z^2=(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta$$
$$=\cos^2\theta+i2\sin\theta\cos\theta-\sin^2\theta$$
$$=\underbrace{(\cos^2\theta-\sin^2\theta)}_{\cos 2\theta}+i\underbrace{(2\sin\theta\cos\theta)}_{\sin 2\theta}$$

Likewise, can prove by induction

$$(r(\cos\theta+i\sin\theta))^n=r^n(\cos(n\theta)+i\sin(n\theta))$$

De Moivre's Thm

Taking $r=1$, $(\cos\theta+i\sin\theta)^n$
$=\cos(n\theta)+i\sin(n\theta)$

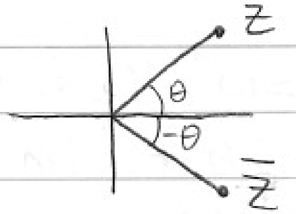
which can be used to deduce triple, quadruple angle formula.

E.g. $n=3$

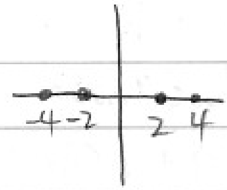
$$\cos(3\theta)+i\sin(3\theta)=(\cos\theta+i\sin\theta)^3$$
$$=\cos^3\theta+3\cos^2\theta\, i\sin\theta+3\cos\theta\, i^2\sin^2\theta+i^3\sin^3\theta$$
$$=\underbrace{(\cos^3\theta-3\cos\theta\sin^2\theta)}_{\cos(3\theta)}+i\underbrace{(3\cos^2\theta\sin\theta-\sin^3\theta)}_{\sin(3\theta)}$$

What about negative multiples?

Hint: $\cos(-\theta)+i\sin(-\theta)$
$=\cos\theta-i\sin\theta=\overline{\cos\theta+i\sin\theta}$



We've seen if $z = r(\cos\theta + i\sin\theta)$
then $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$
What about the roots?



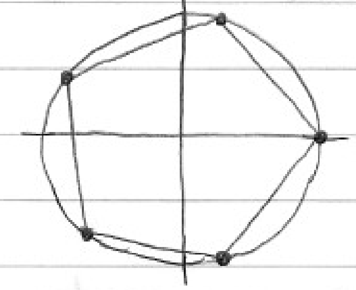
$\sqrt{4} = \pm 2$
$\sqrt{-4} = \pm 2i$

if $z^n = r(\cos\theta + i\sin\theta)$, what is z?
let $z = s(\cos\phi + i\sin\phi)$, then
De Moivre's $\Rightarrow z^n = s^n(\cos(n\phi) + i\sin(n\phi))$
Easy to check $|zw| = |z||w|$
and hence by induction $|z^n| = |z|^n$ (exercise)
hence $r = |z^n| = |z|^n = s^n$
Since r, s real + nonnegative #s,
$s = \sqrt[n]{r} = r^{1/n}$
Hence $\cancel{r}(\cos\theta + i\sin\theta) = \cancel{s^n}(\cos(n\phi) + i\sin(n\phi))$
$\Rightarrow \cos\theta = \cos(n\phi)$ and $\sin\theta = \sin(n\phi)$
$\Rightarrow \theta = n\phi \pmod{2\pi}$
$\Rightarrow \theta/n = \phi \pmod{\frac{2\pi}{n}}$
I.e. $\phi = \theta/n,\ \theta/n + 2\pi/n,\ \theta/n + 4\pi/n,$
$\theta/n + 6\pi/n$, etc.

unity $\equiv 1$

E.g. what if $z^n = 1$?
I.e. $r = 1$, $\theta = 0$
then $s = 1^{1/n} = 1$, $\phi = 0, 2\pi/n, 4\pi/n, 6\pi/n, 8\pi/n$, etc.

E.g. here are the $\sqrt[5]{1}$ in $\mathbb{C}$:

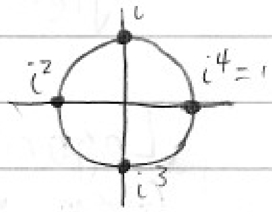


$\cos\theta + i\sin\theta$
$\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$
$\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$
$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$
$\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

Generally, any $r(\cos\theta + i\sin\theta) \neq 0$ will have exactly n nth roots, on a regular n-gon inscribed in a circle of radius $r^{1/n}$:

$$r^{1/n}\left(\left(\cos\frac{\theta + 2\pi j}{n}\right) + i\left(\sin\frac{\theta+2\pi j}{n}\right)\right),\ j = 0 \text{ to } n-1$$

Complex exponentials



what is e^{a+bi}?
Expect: $e^{a+bi} = e^a e^{bi}$
taylor series for e^x centered at 0 is:
$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5! \ldots$

$$e^{ib} = 1 + ib + \frac{i^2b^2}{2!} + \frac{i^3b^3}{3!} + \frac{i^4b^4}{4!} + \frac{i^5b^5}{5!} + \frac{i^6b^6}{6!} \ldots$$

$= 1 \quad - b^2/2! \quad + b^4/4! \quad - b^6/6!$

$+ i(\; b \quad - b^3/3! \quad + b^5/5! \quad - \ldots)$

$= \cos b + i \sin b$

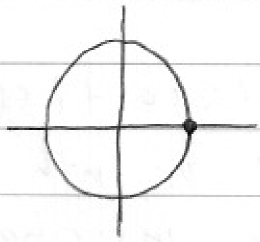
Euler's equation:

$e^{i\theta} = \cos\theta + i\sin\theta$

E.g. $\theta = \pi$

$\Rightarrow e^{i\pi} + 1 = 0$

If $f(t) = e^{it}$ defines a parametric curve in $\mathbb{C}$.



it maps $\mathbb{R}$ onto unit circle wrapping infinitely many times

Polar form can be rewritten

$a + bi = re^{i\theta}$

$\sqrt[n]{re^{i\theta}} = r^{1/n} e^{i\left(\frac{\theta + 2\pi j}{n}\right)}$, $j = 0$ to $n-1$

Logarithms:

Given $re^{i\theta}$, what would $\ln(re^{i\theta})$ mean?

Some $x + iy$ such that

$e^x e^{iy} = e^{x+iy} = re^{i\theta}$

take moduli

$|e^x e^{iy}| = |re^{i\theta}| \Rightarrow |e^x| \cancel{|e^{iy}|} = |r| \cancel{|e^{i\theta}|}$

$\Rightarrow r = e^x \Rightarrow x = \ln(r)$

$\mathbb{R}$ - real
$\mathbb{C}$ - complex
$\mathbb{Z}$ - integer
$\mathbb{Q}$ - rational #s

$\Rightarrow \cancel{r}e^{iy} = \cancel{r}e^{i\theta}$
$\Rightarrow e^{iy} = e^{i\theta}$
$\Rightarrow y \equiv \theta \pmod{2\pi}$
(ie. $y-\theta$ is an integer multiple of 2π)
I.e. $y = \theta + 2\pi j$, $j \in \mathbb{Z}$
that is,
$\ln(re^{i\theta}) = \ln(r) + i(\theta + 2\pi j)$,
where j could be any integer!
Every (nonzero) complex # has an infinite # of lns.