Mathematics V1205y
Calculus IIIS/IVA

Answers to Midterm Exam #2
April 17, 2000
9:10–10:35 am

1. True: since line integrals are independent of path, $\mathbf{F}$ must be a gradient, and the curl of a gradient vanishes.

2. True by the divergence theorem, since the divergence of $\mathbf{F}$ vanishes.

3. False: let $f(x,y,z)=x^2$; then $\nabla\cdot(\nabla f)=2$.

4. By Green's theorem, this equals $\iint_D(\partial Q/\partial x-\partial P/\partial y)\,dA$, where D is the interior of the triangle, and $\partial Q/\partial x-\partial P/\partial y=2x-x=x$. This equals the iterated integral $\int_0^1\int_{3x}^3 x\,dy\,dx=\int_0^1(3x-3x^2)\,dx=3/2-1=1/2$.

5. The curve bounds the portion S of the paraboloid enclosed by the cylinder, that is, where $z=x^2-y^2$ and $2x^6+3y^2\le 10$. By Stokes's theorem, $\oint_C\mathbf{F}\cdot d\mathbf{r}=\iint_S\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$. So it suffices to find a and b for which the curl vanishes. The curl is $\nabla\times\mathbf{F}=(b+\cos y-\cos y,0,-1-a)$, so $a=-1$, $b=0$.

6. Parametrize S by $D=$ the unit disk, $\mathbf{r}(u,v)=(u,v,1-u^2-v^2)$. Then $\mathbf{r}_u\times\mathbf{r}_v=(1,0,-2u)\times(0,1,-2v)=(2u,2v,1)$ points upward, so

$$\begin{aligned}
\iint_S\mathbf{F}\cdot d\mathbf{S} &= \iint_D(\sqrt{u^2+v^2},0,-2e^{1-u^2-v^2})\cdot(2u,2v,1)\,du\,dv\\
&= \iint_D(2u\sqrt{u^2+v^2}-2e^{1-u^2-v^2})\,du\,dv\\
&= \int_0^{2\pi}\int_0^1 2r\cos\theta\,r\,r\,dr\,d\theta+\int_0^{2\pi}\int_0^1-2e^{1-r^2}r\,dr\,d\theta\\
&= \int_0^1 r^3\,dr\int_0^{2\pi}\cos\theta\,d\theta+2\pi\int_0^1-2re^{1-r^2}\,dr=0+2\pi(e^{1-r^2})_0^1=2\pi(1-e).
\end{aligned}$$

7. The curl is $e^{xyz}(x+x^2yz-x-x^2yz,-y-xy^2z+y+xy^2z,z+xyz^2-z-xyz^2)=\mathbf{0}$. Since $\mathbf{F}$ is defined on all of $\mathbf{R}^3$, which is convex, it follows that $\mathbf{F}$ is conservative. To find the value of the potential at $\mathbf{v}=(x,y,z)$, let $\mathbf{r}(t)=t\mathbf{v}$, $t\in[0,1]$, be the straight-line path $C_\mathbf{v}$ from $\mathbf{0}$ to $\mathbf{v}$. Then $\int_{C_\mathbf{v}}\mathbf{F}\cdot d\mathbf{r}=\int_0^1(t^2yze^{t^3xyz},t^2xze^{t^3xyz},t^2xye^{t^3xyz}+\cos tz)\cdot(x,y,z)\,dt=\int_0^1(3xyzt^2e^{t^3xyz}+z\cos tz)\,dt=(e^{t^3xyz}+\sin tz)_{t=0}^{t=1}=e^{xyz}+\sin z$. Alternative: integrate $\partial f/\partial x=yze^{xyz}$ with respect to x to find $f=e^{xyz}+C(y,z)$. Then differentiate with respect to y and z, and compare with the other two components of $\mathbf{F}$, to find $\partial C/\partial y=0$, $\partial C/\partial z=\cos z$, so $C=\sin z+K$.