Mathematics V1202
Calculus IV
Practice Final Exam

1. Convert the integral $\int_0^2 \int_{-\sqrt{4-x^2}}^0 \frac{xy}{\sqrt{x^2+y^2}}\, dy\, dx$ to polar coordinates and evaluate it.

2. Exploit symmetry to evaluate $\iint_D (\sin^{35} x + e^y + ye^{y^4})\, dy\, dx$, where $D = [-1,1] \times [-1,1]$. Explain your reasoning clearly!

3. Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (1 + \tan^2 x, x^2 + e^y)$, and C is the counterclockwise boundary of the plane region E enclosed by $y = \sqrt{x}$, $x = 1$, and $y = 0$.

4. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z) = (xz, yz, z^2)$, and S is the portion of the sphere $x^2+y^2+z^2 = 1$ in the first octant $(x,y,z \geq 0)$, oriented outward.

5. Let $\mathbf{F}(x,y,z) = ((z-5)^{72} - yz, xz, xy)$. Compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.

6. Let $\mathbf{r} = (x,y,z)$. Compute the gradient of $\ln |\mathbf{r}|$. Express your answer in terms of $\mathbf{r}$.

7. Find a function g such that $\mathbf{F} = \nabla g$ and use it to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
$\mathbf{F}(x,y) = (e^{2y}, 1 + 2xe^{2y})$; $\quad C = \{(te^t, 1+t) : t \in [0,1]\}$.

8. Suppose that $\mathbf{F}$ and $\mathbf{G}$ are vector fields, defined and with continuous partials on all of $\mathbf{R}^3$, which have the same curl: $\nabla \times \mathbf{F} = \nabla \times \mathbf{G}$. Also suppose that C and D are piecewise smooth curves in $\mathbf{R}^3$ having the same initial point and the same terminal point. Show that $\int_C \mathbf{F} \cdot d\mathbf{r} + \int_D \mathbf{G} \cdot d\mathbf{r} = \int_C \mathbf{G} \cdot d\mathbf{r} + \int_D \mathbf{F} \cdot d\mathbf{r}$. State clearly what domains of integration you use.

9. Clearly, concisely, completely, and correctly state the Cauchy Integral Theorem.

10. Determine all the cube roots of $-i$. Express them in rectangular form.

11. Find a nonzero harmonic function $f(x,y)$ that vanishes at $(1,0)$, $(-1,0)$, $(0,1)$, and $(0,-1)$.

12. If a complex function $g(z)$ is differentiable at all $z \in \mathbf{C}$, and its real and imaginary parts are equal, does it have to be constant? Why or why not?

13. Express $e^z/(z^2 - z)$ as a sum of two terms of the form $f(z)/(z - z_0)$, with $f(z)$ holomorphic. Use this to evaluate

$$\oint_C \frac{e^z\, dz}{z^2 - z},$$

where C is the ellipse $x^2/9 + y^2 = 1$, oriented counterclockwise.