Mathematics V1202
Calculus IV
Answers to Practice Final

1. The region of integration is the intersection of the disk of radius 2 with the 4th quadrant, so using $x = r\cos\theta$, $y = r\sin\theta$, and $dx\,dy = r\,dr\,d\theta$, we get

$$\int_{3\pi/2}^{2\pi}\int_0^2 \frac{(r\cos\theta)(r\sin\theta)}{r}\,r\,dr\,d\theta = \left(\int_0^2 r^2\,dr\right)\left(\int_{3\pi/2}^{2\pi}\cos\theta\sin\theta\,d\theta\right) = \tfrac{8}{3}\cdot -\tfrac{1}{2} = -\tfrac{4}{3}.$$

2. The domain D is symmetric under the reflection taking (x, y) to $(-x, y)$, and the first term in the integrand is an odd function of x, so its integral vanishes. Likewise, D is symmetric under the reflection taking (x, y) to $(x, -y)$, and the third term in the integrand is an odd function of y, so its integral vanishes. All that remains is $\iint_D e^y\,dy\,dx = e^y|_{-1}^1 = e - 1/e$.

3. By Green's theorem, this $= \iint_E (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\,dA = \iint_E 2x\;dx\,dy = \int_0^1\int_{y^2}^1 2x\;dx\,dy = \int_0^1 (1 - y^4)\,dy = 4/5$.

4. Parametrize S as $\{(u, v, \sqrt{1 - u^2 - v^2}) \mid (u, v) \in D\}$, where D is the portion of the unit disk in the first quadrant. Then $\mathbf{r}_u = (1, 0, -u/s)$, $\mathbf{r}_v = (1, 0, -v/s)$, and $\mathbf{r}_u \times \mathbf{r}_v = (u/s, v/s, 1)$, where s is short for $\sqrt{1 - u^2 - v^2}$. This indeed points outward since the last component is positive. The surface integral is then $\iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{r}_u \times \mathbf{r}_v\,du\,dv = \iint_D (us, vs, 1 - u^2 - v^2) \cdot (u/s, v/s, 1)\,du\,dv = \iint_D 1\,du\,dv = \frac{\pi}{4}$.

5. This forces you to use Stokes, since the surface integral is cumbersome to do directly. According to Stokes it equals $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$. Let $\mathbf{r}(t) = (2\cos t, 2\sin t, 5)$ for $t \in [0, 2\pi]$; then the line integral equals

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} (-10\sin t, 10\cos t, 4\cos t\sin t)\cdot(-2\sin t, 2\cos t, 0)\,dt = 40\pi.$$

6. $\ln|\mathbf{r}| = \ln\sqrt{x^2 + y^2 + z^2}$, so $\partial/\partial x \ln|\mathbf{r}| = x/(x^2 + y^2 + z^2)$ using the quotient rule and $\partial/\partial x \ln f(x) = (\partial f/\partial x)/f$. Likewise for y and z, so $\nabla \ln|\mathbf{r}| = (x, y, z)/(x^2 + y^2 + z^2) = \mathbf{r}/|\mathbf{r}|^2$.

7. Following the Stewart method, integrate both sides of $\partial g/\partial x = e^{2y}$ to get $g(x, y) = xe^{2y} + C(y)$; then $1 + 2xe^{2y} = \partial g/\partial y = 2xe^{2y} + C'(y)$, so $C'(y) = 1$ and $C(y) = y + K$. Let's take $K = 0$; then $g(x, y) = xe^{2y} + y$. By the FTLI, $\int_C \mathbf{F} \cdot d\mathbf{r} = g(e, 2) - g(0, 1) = e^5 + 2 - 1 = e^5 + 1$.

8. Let $E = C \cup -D$: since C and D have the same endpoints, E is a closed curve. Let $\mathbf{H} = \mathbf{F} - \mathbf{G}$: then $\nabla \times \mathbf{H} = 0$ and, since $\mathbf{H}$ is defined on all of $\mathbf{R}^3$, it must be conservative. Then by the FTLI,

$$0 = \oint_E \mathbf{H}\cdot d\mathbf{r} = \int_C \mathbf{H}\cdot d\mathbf{r} - \int_D \mathbf{H}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r} - \int_C \mathbf{G}\cdot d\mathbf{r} - \int_D \mathbf{F}\cdot d\mathbf{r} + \int_D \mathbf{G}\cdot d\mathbf{r};$$

now add the terms with minus signs to both sides.

Alternate method: let S be the surface bounded by E. Then by Stokes's theorem $\oint_E \mathbf{H} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{H} \cdot d\mathbf{S} = \iint_S 0 \cdot d\mathbf{S} = 0$. Less preferable since it's not so obvious why or whether the surface S exists.

9. Let $f(z)$ be a complex function, holomorphic on an open set containing a plane region D. Then $\oint_{\partial D} f(z)\, dz = 0$.

10. Since $-i = e^{3\pi i/2}$, its cube roots are $e^{i\theta}$ for $\theta \equiv \pi/2 \pmod{2\pi/3}$, which means $\theta = \pi/2, \pi/2 + 2\pi/3, \pi/2 + 4\pi/3$, that is, $90°, 210°, 330°$. In rectangular coordinates, $e^{i\theta} = \cos\theta + i\sin\theta$, which for these three angles is i, $-\sqrt{3}/2 - i/2$, and $\sqrt{3}/2 - i/2$ respectively.

11. The only way we know how to find harmonic functions is as the real or imaginary parts of holomorphic functions. What kind of holomorphic function can be arranged to vanish at the four points, which in complex language are ± 1 and $\pm i$? How about a polynomial with these as its roots, namely $(z-1)(z+1)(z-i)(z+i) = (z^2-1)(z^2+1) = z^4 - 1$? Express this as $(x+iy)^4 - 1$, expand with the binomial theorem, and take the real part to get $x^4 - 6x^2y^2 + y^4 - 1$. Or take the imaginary part to get $4x^3y - 4xy^3$.

By sheer guesswork, you could also have come up with the simple answer xy. This is the imaginary part of $z^2/2$, which doesn't vanish at ± 1 and $\pm i$, but is real there.

12. Yes, it does: for if $f = u + iv$ with $u = v$, then by the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y = \partial u/\partial y = -\partial v/\partial x = -\partial u/\partial x$, so all partials vanish and hence u and v are both constant.

13. Use partial fractions: $z^2 - z = (z-1)z$, so if $A/(z-1) + B/z = 1/(z^2 - z)$, we have $zA + (z-1)B = 1$, hence $A = 1$, $B = -1$. Multiplying by e^z gives $e^z/(z^2 - z) = e^z/(z-1) - e^z/z$, which are both of the desired form with $z_0 = 1$ and 0, respectively. Since both of these points are enclosed by the given ellipse, by the Cauchy Integral Formula

$$\oint_C \frac{e^z\, dz}{z^2 - z} = \oint_C \frac{e^z\, dz}{z-1} - \oint_C \frac{e^z\, dz}{z} = 2\pi i(e - 1).$$