Fundamental theorem of Calculus for line integrals:

If $\mathbf{F} = \nabla f$ is the gradient of a function, then $f$ is called the potential of $\mathbf{F}$.

In this case:

$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$

Ex: In the case of gravity

$\mathbf{F}(x, y, z) = -\frac{1}{r^3} \hat{r}$, we have $f = \frac{1}{r^2} = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$

$\nabla f = -\frac{1}{2} \left( \frac{2x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{2y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{2z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right)$

$\mathbf{F}(x, y, z) = \vec{F}(x, y, z)$

$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
Why is this true?

\[ \int_C \nabla \Phi \cdot dr = \int_a^b \left( \frac{\partial \Phi}{\partial x} \frac{dx}{dt} + \frac{\partial \Phi}{\partial y} \frac{dy}{dt} + \frac{\partial \Phi}{\partial z} \frac{dz}{dt} \right) dt \]

= \frac{df}{dt} \text{ by Chain rule}

= \int_a^b \frac{df}{dt} (\Phi(t)) dt

by FTC = f(\Phi(b)) - f(\Phi(a))

- This means that to compute the line integral of F, we just need to know the value of the potential \( \Phi \) at the two endpoints.

- In particular, any two curves with the same endpoints will have the same line integral.

- In physics, \( f \) is minus the potential energy, and \( F = -\nabla V \) is the potential energy.

- When this is the case, it also implies the line integral over closed curves is always 0.
\( \mathbb{E} \): i) \( \vec{F}(x,y) = \vec{r} \), \( C \) = parabola \( y = x^2 \) from \( x = 0 \) to \( x = 1 \) followed by a line segment from \((1,1)\) to \((2,2)\).

Notice: \( \vec{F}(x,y) = \nabla \frac{1}{2} \left( \nabla \left( \frac{1}{2} \right) \right) = \nabla \left( \frac{x^2 + y^2}{2} \right) \)

so \( \int_C \vec{F} \cdot d\vec{r} = \frac{x^2 + y^2}{2} \bigg|_{x=2, y=2}^{x=0, y=0} - \frac{x^2 + y^2}{2} \bigg|_{x=0, y=0} = 4 \)

ii) What if \( \vec{F}(x,y) = (y, x) \) over the same curve \( C \)?

Notice: \( \vec{F}(x,y) = \nabla (xy) \)

so \( \int_C \vec{F} \cdot d\vec{r} = xy \bigg|_{x=2, y=2}^{x=0, y=0} - xy \bigg|_{x=0, y=0} = 4 \).

Q: Can \( \vec{F} = -y\vec{i} + x\vec{j} \) be written as \( \nabla f \) for some \( f \)?

No, because the integral along the red curve is not 0.
**Conservation of energy:** Suppose $\mathbf{F} = -\nabla V$ is a conservative force.

$$\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t) \quad \text{Newton's 2nd law}$$

$$W = \int_{C} \mathbf{F} \cdot d\mathbf{r} = \int_{a}^{b} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_{a}^{b} m \mathbf{r}''(t) \mathbf{r}'(t) \, dt$$

$$= \frac{m}{2} \left[ \frac{d}{dt} \left( \mathbf{r}'(t) \cdot \mathbf{r}'(t) \right) \right]_{a}^{b}$$

$$= \frac{m}{2} \int_{a}^{b} \frac{d}{dt} \left| \mathbf{r}'(t) \right|^2 \, dt$$

$$= \frac{m}{2} \left| \mathbf{r}'(b) \right|^2 - \frac{m}{2} \left| \mathbf{r}'(a) \right|^2$$

$$= K(b) - K(a)$$

So work done = final kinetic energy - initial kinetic energy

by FTC the line integral work done = $V(\mathbf{r}(b)) - V(\mathbf{r}(a))$

- initial potential energy
- final potential energy

$\Rightarrow K(b) - K(a) = V(a) - V(b) \Rightarrow K(a) + V(a) = K(b) + V(b)$. 
When is a vector field conservative? (When is $\vec{F}$ a gradient?)

When $\vec{F} = \nabla f$ is a gradient vector field, we say $\vec{F}$ is conservative.

Q: Which vector fields are conservative?

In $\mathbb{R}^1$, all vector fields are conservative.

$\vec{F}(x) = g(x) \hat{i}$, then set $F(x) = \int_0^x g(t) \, dt \quad \text{ FTC } \Rightarrow \nabla f = \vec{F}$

In $\mathbb{R}^2$, we've seen:

Conservative $\implies$ Path independent property

(so for example $y^2 \hat{i} + x \hat{j}$ is not conservative)

In fact path independent property $\implies$ Conservative

set $t(x,y) = \int_{(x_0,y_0)}^{(x,y)} \vec{F}(x,y) \, dx$

then $t(x,y)$ is well-defined and

$\left( \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y} \right) = \left( \frac{\partial}{\partial x} \int_{(x_0,y_0)}^{(x,y)} P \, dx + Q \, dy \bigg|_{(x_0,y_0)} \right) \frac{\partial}{\partial y} \int_{(x_0,y_0)}^{(x,y)} P \, dx + Q \, dy \bigg|_{(x_0,y_0)} \right)$

by FTC

$\implies (P, Q) = \vec{F}$
But the path independent property is hard to check!

**Criterion 2:**

Now if \( \mathbf{F} = \langle P, Q \rangle = \langle \frac{2x}{x}, \frac{2y}{y} \rangle \)

then \( \frac{\partial P}{\partial y} = \frac{2y}{x} = \frac{\partial Q}{\partial x} = \frac{2x}{y} \),

so \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

(This also implies \( y^2 + x^2 \) is not conservative since \( \frac{\partial P}{\partial y} = 2y, \frac{\partial Q}{\partial x} = 1 \).)

**Def.** A domain \( D \) is simply connected if it is connected and every loop contracts to a point.

**Thm.** If \( \mathbf{F} = P \mathbf{i} + Q \mathbf{j} \) is a vector field defined on a simply connected domain \( D \), and \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \), then \( \mathbf{F} \) is conservative.

This theorem is a consequence of Green's theorem, which we'll discuss later.
i) Is \( \vec{F}(x,y) = (x-y)^2 + (x-2)^2 \) conservative?

\[
\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1 \quad \text{so No!}
\]

\[
\frac{\partial P}{\partial y} = 2x, \quad \frac{\partial Q}{\partial x} = 2x, \quad \mathbb{R}^2 \text{ is simply connected}
\]

so Yes! \( \Rightarrow \vec{F} = \nabla f \) for some function \( f \).

b) Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) when \( \vec{r}(t) = e^t \sin t \vec{i} + e^t \cos t \vec{j} \) for \( 0 \leq t \leq \pi \)

Q: What is \( f \)?

Theorem doesn't tell us, have to find it!

Since \( \nabla f = \vec{F} \) \( \Rightarrow \frac{\partial f}{\partial x} = 3 + 2xy \}

\[
\frac{\partial f}{\partial y} = x^2 - 3y^2
\]

\[
\frac{\partial f}{\partial x} = 3 + 2xy
\]

\[
\Rightarrow \int_0^x \frac{\partial f}{\partial x} (t,y) \, dt = \int_0^x (3 + 2ty) \, dt
\]

\[
f(x,y) - f(0,y) = 3x + x^2 y
\]
we also know \( \frac{\partial f}{\partial y} = x^2 - 3y^2 \). So

\[
x^2 + \frac{\partial f}{\partial y}(0, y) = x^2 - 3y^2
\]

\[
y = \frac{\partial f}{\partial y}(0, y) = -3y^2
\]

\[= f(0, y) = -y^3 + K.
\]

so \( f(x, y) = 3x + x^2 y - y^3 + K. \)

\[
y \int_{C} \vec{F} \cdot d\vec{r} = f(e^\pi \sin \theta, e^\pi \cos \theta) - f(e^0 \sin 0, e^0 \cos 0)
\]

\[
= f(0, e^\pi) - f(0, 1)
\]

\[
= -(-e^\pi)^3 + 1
\]

\[
= e^{3\pi} + 1
\]