Spherical cood: \((\rho, \Theta, \phi)\) \(\rho \geq 0, \ 0 \leq \phi \leq \pi, \ 0 \leq \Theta \leq 2\pi\)

\[
\begin{align*}
X &= \rho \sin \phi \cos \Theta \\
y &= \rho \sin \phi \sin \Theta \\
z &= \rho \cos \phi \\
\rho^2 &= x^2 + y^2 + z^2 \\
\tan \Theta &= \frac{y}{x} \\
\cos \phi &= \frac{z}{\sqrt{x^2 + y^2}}
\end{align*}
\]

Change of variables formula to spherical cood

\(E = \{a \leq x \leq b, \ a \leq \Theta \leq \beta, \ c \leq \phi \leq d\}\) is a spherical wedge

\[
\iiint_E f(x, y, z) \, dV = \int_a^b \int_c^d \int_a^b f\left(\rho \sin \phi \cos \Theta, \ \rho \sin \phi \sin \Theta, \ \rho \cos \phi\right) \rho^2 \sin \phi \, d\rho \, d\phi \, d\Theta
\]

\[\Rightarrow \Delta V \sim (\rho \Delta \phi)(\rho \sin \phi \Delta \Theta) \, \Delta \rho\]
Examples: (of integration in spherical coord)

i) Compute the volume of the solid between $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 2$

$$
E = \left\{ x^2 + y^2 + (z - \frac{1}{2})^2 \leq \frac{1}{4} \right\}
$$

$$
V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{\cos \phi}} r^2 \sin \phi \, dr \, d\phi \, d\theta
$$

$$
= \frac{2\pi}{3} \int_0^{\pi/4} (\cos \phi)^{3/2} \sin \phi \, d\phi
$$

$$
= \frac{2}{3} \left[ -\frac{1}{4} \cos^4 \phi \right]_0^{\pi/4}
$$

$$
= \frac{\pi}{9}
$$
ii) Compute the volume of the tear drop \( v = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\cos \phi} r^2 \sin \phi \, dr \, d\phi \, d\theta \).

\[
\begin{align*}
\iiint_{E} dv &= \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{4}} \int_{0}^{\cos \phi} r^2 \sin \phi \, dr \, d\phi \, d\theta \\
&= \frac{2\pi}{3} \int_{0}^{\frac{\pi}{4}} (\sin \phi) (\cos^2 \phi \sin^2 \phi) \, d\phi \\
&= \frac{2\pi}{3} \int_{0}^{\frac{\pi}{4}} (\sin \phi) \left( \cos^2 \phi - \sin^2 \phi \right) \, d\phi \\
&= \frac{2\pi}{3} \int_{0}^{\frac{\pi}{4}} (\sin \phi) \left( 2 \cos^2 \phi - 1 \right) \, d\phi \\
&= \frac{2\pi}{3} \int_{-\cos \phi}^{1} \left( 2u^2 - 1 \right) \, du \\
&= \frac{2\pi}{3} \int_{-1}^{1} \left( 8u^6 - 12u^4 + 6u^2 - 1 \right) \, du \\
&= \frac{2\pi}{3} \left[ \frac{8u^7}{7} - \frac{12u^5}{5} + \frac{6u^3}{3} - u \right]_{u=-1}^{u=1} \\
&= \frac{2\pi}{3} \left( \frac{8}{7} - \frac{12}{5} + 2 - 1 - \frac{1}{7\sqrt{2}} + \frac{3}{5\sqrt{2}} \right) \\
&= \frac{2\pi}{105} \left( 8\sqrt{2} - 9 \right)
\end{align*}
\]
8. Change of variables

Recall: In 1D, some times we can use substitution to simplify an integral

\[ \int_{a}^{b} f(x) \, dx = \int_{c}^{d} f(x(u)) \frac{dx}{du} \, du \]

where \( x(a) = a, \ x(d) = b \).

In 2D, we also learned that sometimes it's useful to change integrals into polar coord to simplify them

\[ \iint_{R} f(x,y) \, dx \, dy = \iint_{S} f(r\cos \theta, r\sin \theta) \, r \, dr \, d\theta \]

where \( x = r\cos \theta, \ y = r\sin \theta \). Where if \( T(r, \theta) = (r\cos \theta, r\sin \theta) \), then \( T(S) = R \).

In general: sometimes we want to transform to more general coordinate systems than polar coord.

Def: A change of variables \((C'-transformation)\) is a map: \(T(u,v) = (x,y)\)

sometimes we write \( x = x(u,v) \), \( y = y(u,v) \)

We say \( T \) is one-to-one if no two points are sent to the same point. Then in fact \( T \) is invertible \((T^{-1} \) exists\).

\[ T^{-1}(x,y) = (u,v) \]

\[ R = \text{image of } S \]
Example: i) \((x, y) = (u^2-v^2, 2uv)\)

What is the image of \([0, 1] \times [0, 1/4]\)?

\[
\begin{aligned}
\begin{array}{c}
(0, 0) \\
(1/4, 0)
\end{array}
\end{aligned}
\xrightarrow{	ext{}}
\begin{aligned}
\begin{array}{c}
x = \frac{x^2-1}{4} \\
y = 1 - \frac{y^2}{4}
\end{array}
\end{aligned}
\]

ii) \((x, y) = (\cos\theta, \sin\theta)\)
What is the image of a rectangle \([0, \pi/2] \times [0, \pi]\)?

\[
\begin{aligned}
\begin{array}{c}
0 \\
\pi/2
\end{array}
\xrightarrow{	ext{}}
\begin{aligned}
\begin{array}{c}
x = 0 \\
y = \pi
\end{array}
\end{aligned}
\end{aligned}
\]

How to integrate:

**Def:** Jacobian of \(T(u, v) = (x, y)\) is the determinant of the matrix of derivations of \(T\):

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}
\]

**Eg:** i) \((x, y) = (an + bv, cu + dv)\)

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc
\]

ii) \((x, y) = (u^2 - v^2, 2uv)\)

\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix}
2u & -2v \\
2v & 2u
\end{vmatrix} = 4(u^2 + v^2)
\]
\[ (x, y) = (r \cos \theta, r \sin \theta) \]
\[
\begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix} = r (\cos^2 \theta + \sin^2 \theta) = r
\]

**Change of variables formula in 2D**

Now suppose \( T(u, v) = (x, y) \) is a one-to-one \( C^1 \)-transformation which takes a region \( S \) to \( R \) \((T(S) = R)\), then

\[
\iint_R f(x, y) \, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, du \, dv
\]

\[= \Delta A \]

If \( \Delta u, \Delta v \) are very small, the \( T \) is “almost a linear map”

\[
\Delta A \approx \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} \Delta u \Delta v
\]

- We want to use the change of variables formula to change the domain of integration from a “bad shape” to something more natural (e.g., a rectangle)
- Sometimes the integrand is better expressed not as a function of \((x, y)\) but in some other coord. (e.g., \( \sin(x, y) \cos(x^2 - y^2) \)) so set \((u, v) = (x, x^2 - y^2)\)
Ex: 1) Use the same transformation as before \((x,y) = (u^2 - v^2, 2uv)\) to evaluate \(\iint_R y \, dA\) when \(R = \{0 \leq y \leq 2, \ \frac{3}{4} \leq x \leq 1 - \frac{y^2}{4}\}\).

\[\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} 2u & -2v \\ v & 2u \end{vmatrix} = 4(u^2 + v^2)\]

so.

\[\iint_R y \, dA = \left[ \iint_0^1 \left| \frac{\partial (x,y)}{\partial (u,v)} \right| \, dudv \right] \iint_0^1 uu(u^2 + v^2) \, dudv\]

\[= 8 \int_0^1 \left[ \frac{1}{4} u^4 v + \frac{1}{2} u^2 v^3 \right]_{u = 0}^{u = 1} dv = \int_0^1 (2v + 4v^3) \, dv\]

\[= \left[ v^2 + v^4 \right]_0^1 = 2\]