Calculus IV
Midterm 1 Solutions

1. \[
\int_1^2 \int_0^{\sqrt{x}} 2x^2 y \, dy \, dx = \int_1^2 \left[ x^2 y^2 \right]_{y=0}^{\sqrt{x}} \, dx \\
= \int_1^2 x^3 \, dx \\
= \left[ \frac{x^4}{4} \right]_{x=1}^{x=2} \\
= \frac{15}{4}
\]

2. \[
\iint_D \cos(x^3) \, dA = \int_0^1 \int_0^{\sqrt{y}} \cos(x^3) \, dx \, dy \\
= \int_0^1 \int_0^{x^2} \cos(x^3) \, dy \, dx \\
= \int_0^1 x^2 \cos(x^3) \, dx \\
= \left[ \frac{1}{3} \sin(x^3) \right]_{x=0}^{x=2} \\
= \sin(8) \]

3. We denote \( S \) by the graph of the function \( f(x, y) = xy \), over the region \( \{1 \leq x^2 + y^2 \leq 25\} \), so by the formula for surface area, we have

\[
A(S) = \iint_{\{1 \leq x^2 + y^2 \leq 25\}} \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} \, dA
\]

and since \( f(x, y) = xy \), we have \( \frac{\partial f}{\partial x} = y \) and \( \frac{\partial f}{\partial y} = x \), and since the
integral is over an annulus, it’s convenient to use polar coordinates, so
\[
\iint_{\{1 \leq x^2 + y^2 \leq 25\}} \sqrt{1 + x^2 + y^2} \, dA \tag{11}
\]
\[
= \int_0^{2\pi} \int_1^5 \sqrt{1 + r^2} \, dr \, d\theta \tag{12}
\]
\[
= 2\pi \left[ \frac{1}{3} (1 + r^2)^{\frac{3}{2}} \right]_{r=1}^{r=5} \tag{13}
\]
\[
= \frac{2\pi}{3} \left( 26^{\frac{3}{2}} - 2^{\frac{3}{2}} \right) \tag{14}
\]

4. We can write the region tetrahedron \( E \) as the region \( \{0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\} \), so we have
\[
\iiint_E (1 - z^2) \, dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 - z^2) \, dz \, dy \, dx \tag{15}
\]
\[
= \int_0^1 \int_0^{1-x} ((1 - x - y) - \frac{(1-x-y)^3}{3}) \, dy \, dx \tag{16}
\]
\[
= \int_0^1 \left[ y(1-x) - \frac{y^2}{2} + \frac{(1-x-y)^4}{12} \right]_{y=0}^{y=1-x} \, dx \tag{17}
\]
\[
= \int_0^1 \left( \frac{(1-x)^2}{2} - \frac{(1-x)^4}{12} \right) \, dx \tag{18}
\]
\[
= \left[ \frac{(1-x)^5}{60} - \frac{(1-x)^3}{6} \right]_{x=0}^{x=1} \tag{19}
\]
\[
= \frac{1}{6} - \frac{1}{60} \tag{20}
\]
\[
= \frac{3}{20} \tag{21}
\]
you can also evaluate the integral in a different order, any order would have worked. (In fact, it’s probably easiest to evaluate if we put \( dz \) last and evaluated in the order \( dx \, dy \, dz \) or \( dy \, dx \, dz \))

5. Since \((u, v) = (\frac{y}{x^2}, xy)\), the inverse transformation is given by \((x, y) = ((u)^{\frac{1}{3}}, (uv^2)^{\frac{1}{4}})\), so the Jacobian is given by
\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -\frac{1}{3}v^{\frac{1}{2}}u^{-\frac{1}{2}} & \frac{1}{3}v^{-\frac{1}{2}}u^{-\frac{1}{2}} \\ \frac{1}{3}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{3}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} = -\frac{1}{3u} \tag{22}
\]
now note the equations that defines the boundary of the domain $R$, when expressed in terms variables $(u, v)$ is given by $u = 1$, $u = 4$, $v = 1$ and $v = 5$ so by the change of variable maps a rectangle $[1, 4] \times [1, 5]$ onto the region $R$. By the change of variables formula, the area is then given by

$$
\int \int_R dA = \int_1^5 \int_1^4 \frac{1}{3u} \, du \, dv
$$

(23)

$$
= 4 \left[ \frac{1}{3} \log u \right]_{u=1}^{u=4}
$$

(24)

$$
= \frac{4}{3} \log 4
$$

(25)

6. We need to keep track of where each of the three sides of the domain $B$ gets mapped to via the map $(x, y) = (u^2 + v^2, v)$

from this we see that the image of $B$ is the region between the three curves $x = 1$, $y = 0$ and $x = y^2$.

7. Notice that the region of integration is the lower half of the sphere, which we can write as the region $E = \{x^2 + y^2 + z^2 \leq 1, z \leq 0\}$, and in spherical coordinates, this is the region $E = \{\frac{\pi}{2} \leq \phi \leq \pi, 0 \leq \rho \leq 1\}$,
so we rewrite this integral as the integral

\[
\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx = \int_0^{2\pi} \int_\frac{\pi}{2}^{\pi} \int_0^1 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta \\
= \int_0^{2\pi} \int_\frac{\pi}{2}^{\pi} \frac{1}{4} \sin \phi \, d\phi \, d\theta \\
= 2\pi \left[ \frac{1}{4} \cos \phi \right]_{\phi=\frac{\pi}{2}}^{\phi=\pi} \\
= \frac{\pi}{2}
\]