Calculus IV, Part 1 Review

The first part of this course so far have been about integration in higher dimension. One thing we’ve focused on is learning to compute multiple integrals, and we’ve learned many ways to do this, which I’ll try to summarize here:

- We can think of the double integral of a function $f(x, y)$ over a domain as the volume of the region between the graph of the function and the $(x, y)$-plane. (In the regions where $f$ is negative, the volume is counted with a negative sign)

- To evaluate an integral, we need to rewrite it as an iterated integral and evaluate the integral with respect to each variable independently.

- Regions that look like $D = \{ a \leq x \leq b, u_1(x) \leq y \leq u_2(x) \}$ are called type I regions, these type of regions can be easily written as an integrated integral.

$$\int \int_D f(x, y) \, dA = \int_a^b \left( \int_{u_1(x)}^{u_2(x)} f(x, y) \, dy \right) \, dx$$

Type II regions are just a type I region with roles of $x$ and $y$ switched and can be written as an interated integral in the opposite order.

- Sometimes regions are neither type I nor type II, but can be split into regions of type I or II. Then we need to evaluate the integral on each region separately.

- Sometimes, integrals can be done with one order of integration much more easily than the other order. In these cases, we need exchange the order of integration to express the integral in the right order.

- In a lot of these problems, the hard part is finding the right bounds of integration; in most cases, it can be very useful to draw a picture.

- Another technique we’ve learned is to switch to polar coordinates, which is the coordinate system $(r, \theta)$ given by a length $r$ and the angle $\theta$ which measures the angle a vector makes with the positive $x$-axis. The transformation rule is given by

$$x^2 + y^2 = r^2 \quad x = r \cos \theta \quad y = r \sin \theta$$

and the formula is given by

$$\int \int_D f(x, y) \, dA = \int_{\alpha}^{\beta} \int_{a}^{b} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta$$

if $D = \{ a \leq r \leq b, \alpha \leq \theta \leq \beta \}$. Polar coordinate are very useful when your domain has some rotational symmetry (eg, a ball, a half ball, an annulus), or when your integrand can more easily be expressed as a function of $(r, \theta)$. (eg. $\cos(x^2 + y^2)$, $e^{x^2+y^2}$, $\frac{x}{\sqrt{x^2+y^2}}$)
• One application that can be done with integration is to compute the surface area of
a graph of a function. Let \( S \) be the graph of \( f(x, y) \) over a domain \( D \), then the area
of \( S \) is given by the formula

\[
A(S) = \iint_D \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2} \, dA
\]

this can be used to compute things like the surface area of a sphere.

• Triple integrals work in exactly the same way as double integrals, but evaluating them
can be more complicated as there are more variables lying around.

• To evaluate triple integrals, we also have to write them as an interated integral. Just
as is the case with double integrals, one of the main techniques to evaluate triple
integrals is to switch order of integration; sometimes one order is just a lot easier to
evaluate than the other one. For triple integrals, this is much harder partly due to our
limited capability to draw and visualize 3D figures. One trick is to try to swapping
two of the integrals each time and working out what the new bounds of integration
should be.

• Sometimes if the domain or integrand has 1 degree of rotational symmetry, it can
be useful to change to cylindrical coordinate, which is the coordinate system \( (r, \theta, z) \)
which is related to the usual coordinate system by

\[
\begin{align*}
  r^2 &= x^2 + y^2 & x &= r \cos \theta \quad y = r \sin \theta \quad z = z
\end{align*}
\]

This is really just the coordinate system we get by fixing one of the axis \( z \) and using
polar coordinate on the other two coordinates. So the change of variable formula is
given by

\[
\iiint_D f(x, y, z) \, dV = \int_a^b \int_\alpha^\beta \int_c^d f(r \cos \theta, r \sin \theta, z) r \, dr \, d\theta \, dz
\]

if \( D = \{ a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d \} \).

• When the domain has some spherical symmetry, it is useful to use spherical coordi-
nates \( (\rho, \phi, \theta) \) given by

\[
\begin{align*}
  x &= \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi
\end{align*}
\]

and the change of variable formula is given by

\[
\iiint_E f(x, y, z) \, dV = \int_\alpha^\beta \int_\gamma^\delta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

if \( E = \{ a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \delta \} \).
• All the special coordinate system we’ve learned can be encompassed in one single technique called change of variables. Change of variables says suppose \( T(u,v) = (x,y) \) is an one-to-one transformation (invertible) transformation from a region \( S \) in the \((u,v)\)-plane to the region \( R \) in the \((x,y)\)-plane, then

\[
\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dudv
\]

The main thing in the formula to keep note of is the expression for the new area element \( dA_{x,y} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dudv \), where \( \frac{\partial(x,y)}{\partial(u,v)} \) is the determinant of the matrix of partial derivatives. This term keeps tracks of essential how the area changes when one changes from \((x,y)\) coordinates to \((u,v)\) coordinates via the transformation \( T \).

• The change of variables formula in 3D is exactly the same except now we have to take the determinant of a 3x3 matrix in the expression of the Volume element in the new coordinate system.

• There are two main ways we would like to apply the change of variable formula. Sometimes we want to simplify the domain of integration \( R \) by finding a transformation \( T \) that maps a simpler region \( S \) (eg. a square, a ball ...) onto the region \( R \), then we can use change of variables formula to re-express the integral over \( R \) as an integral over the simpler region \( S \).

• The second way we want to use the change of variable formula is to simplify the expression in the integrand. For example, if the integrand looks like \( \cos\left(\frac{x+y}{x-y}\right) \), it is not so easy to find an antiderivative of this with respect to any of the two variables. However, if we write \( (u,v) = (x+y,x-y) \), then the expression simplify to \( \cos\left(\frac{2u}{2}\right) \), and we can see very easily what the antiderivative of this new expression should be.

Now we’ve spent a lot of time learning techniques to compute integrals, but it is also very important to have a good conceptual understanding of what the integrals represent and why the formulas are the way they are. Having a good conceptual understanding will help you develop a clearer picture in your mind, and allows you to think about the subject in a more flexible way. This is something that’s very useful, especially if you will be seeing this material later on in other courses in different contexts. A better conceptual understanding can also help one when solving problems, as often one of the hard aspect of solving a problem is in the set-up, and having a clear picture in mind can help one find the most efficient way of setting up a problem.

Practice problems

• (15.2.48,49,50) Switch the order of integration on the following integrals

\[
\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx \quad \int_{0}^{1} \int_{0}^{\arctan x} f(x,y)\,dy\,dx \quad \int_{1}^{1} \int_{0}^{\log x} f(x,y)\,dy\,dx
\]
• (15.2.43) Find the volume of the solid enclosed by \( z = 1 - x^2 - y^2 \) and \( z = 0 \).
• (15.3.37) Find the average value of \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \) in the region \( a^2 \leq x^2 + y^2 \leq b^2 \).

Evaluate the following integrals:
\[
\begin{align*}
\int\int_D x^2 \, dA & \quad D = \{|x| + |y| \leq 1\}, \\
\int\int_D xy^2 \, dA & \quad D = \{|x|^2 + |y|^2 \leq 1\}
\end{align*}
\]
\[
\int\int_D \frac{1}{1 + x^2} \, dA \quad D \text{ is a triangle with vertices (0, 0), (1, 1), and (0, 1)}
\]
\[
\int_0^1 \int_0^1 e^{\max(x^2, y^2)} \, dx \, dy
\]

• (15.3.18) Find the area of the region inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 3 \cos \theta \).

Let \( R \) be the region in the first quadrant of the \( xy \) plane which is enclosed by \( y = \sqrt{x}, \ x = 0 \) and \( y = 1 \). Compute the volume of the solid which is bounded above by \( z = xe^{x^2/y^2} \) and has \( R \) as its base.

Let \( R \) be the region enclosed by \( y = x^2 \) and \( y = 2x + 3 \), setup the double integral that represents the area of this region.

• (15.8.12) Sketch the region \( \{1 \leq \rho \leq 2, \pi/2 \leq \phi \leq \pi\} \).

• (15.8.28) Find the average distance from a point in a ball of radius \( a \) (in 3D) to its center.

• (15.8.29)
  1. Find the volume of the solid that lies above the cone \( \phi = \pi/3 \) and the sphere \( \rho = 4 \cos \phi \).
  2. Find the center of mass of this region.

• (15.R.48) Use spherical coordinates to evaluate
\[
\int_{\rho=-2}^{2} \int_{\theta=0}^{\sqrt{4-\rho^2}} \int_{\phi=-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2 + y^2 + z^2} \, dz \, dx \, dy
\]

• (15.R.53) Rewrite the integral
\[
\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx
\]
as an integral in the order \( dx \, dy \, dz \).

• (15.R.56) Use a change of variables formula to find the volume of the region bounded by \( \sqrt{x} + \sqrt{y} + \sqrt{z} = 1 \) and the coordinate planes.