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This is consistent w/ SYZ conj of '96

Plan:
1) LF & Fukaya categories
2) HMS in 1d.
3) SYZ philosophy & construction of minv's
4) stuff in progress / hopes for future.

9/15/16

Lagrangian Floer cohomology

(M,\omega) symplectic
L Lagrangian

thm (Floer): Assume \int \omega = 0 for any disk D w/ \partial D \subset L. Then let \psi \in \text{Ham}(M,\omega) \& \psi(L) \cap L transverse. Then

\text{HF}^\ast(L,\mathbb{Z}) \cong \frac{1}{2} \dim \text{H}^\ast(L,\mathbb{Z}/2).

Note: \psi \in \text{Ham} \Rightarrow \text{area between } L \& \psi(L) \text{ is } 0

not true if \psi \in \text{Symp}

\text{CF}^\ast(L_0, L_1) \text{ is freely generated by } L_0 \wedge L_1

1 \Rightarrow \exists Z \ni \text{HF}^\ast(L,\mathbb{Z}) \cong \text{H}^\ast(L).

If $L_i$ is Ham iso to $L$, then $\text{HF}^\ast(L_0, L_1) = \text{HF}^\ast(L_0, L_1)$. 

\text{bands of disk of pos. area}

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How do we define $\psi_0$?

**Intuition:** Morally, $HF^*$ is the Morse theory of the action functional on (a cover of) the space of paths $\gamma : [0,1] \to (M, \omega, L_1)$ whose cut points are constant paths.

Gradient flow.

We need to translate proof of $\dot{z}^2 = 0$ in Morse theory to the language of pseudoholomorphic curves.

Coeff field: Nakon field over $\text{End}(\mathbb{C}^2, e, \omega)$ $\text{End}(\mathbb{C}^2, e, \omega)$

by $\Lambda_{1k} = \sum a_i \tau^i$ $\lim_{|\tau| \to \infty} a_i = 0$

like a formal power series $\sim \text{just a real exp.}$

This will let $u$ encode symplectic area of trajectories

$$CF^* (L_0, L_1) = \bigoplus_{\gamma \in \gamma_1 \cap \gamma_0} \Lambda_{1k} \mathbb{P}$$

for $\gamma_1 \cap \gamma_0$ intersections

$(M, \omega)$ carries a compatible cs $J$ $\text{Riem}$

satisfying $\text{Jac} = J \text{det}$.

$$\frac{1}{2} \frac{d^2 s}{dt^2} + \frac{1}{2} \frac{\omega(s)}{a^2} = 0$$

$s, t \omega(s, t) \in \mathbb{L}$ $\lim_{s \to \infty} u(s, t) = 0$

$s^2 \omega(s, t) = \text{p}$ $\text{Jac}(\mathbb{L})$

good thinking
Fix the input class \([u] \in \mathfrak{U}_2 (M, L_0, L_1)\).

\[
E(u) = \int_{\mathbb{R} \times [0,1]} \left( \frac{\partial u}{\partial s} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \, ds \, dt
\]

\[
= \int_{\mathbb{R} \times [0,1]} u^* w = \int_{\mathfrak{U}_1} w
\]

Indep of \(u\) w/ input class by Stokes' thm \(w\) is closed.

The \(\mathbb{H}\) linearization of \(DgE\) is Fredholm.
We can calculate its index
\[
\text{ind} (\mathfrak{H}) = \text{ind}([u]).
\]

is the expected dim of the space of sols.

If \(Dg\) is surjective at every sol (generic)
then the space of sols' \(\hat{\mathfrak{M}} (p, q, i, J, [u])\) is
a smooth mfd of dim = \(\text{ind}([u])\).

\[
\mathfrak{M} = \hat{\mathfrak{M}} / \mathbb{R} \quad \text{(trans. in s dir)}
\]

If \(\text{ind}([u]) = 1\), \(\mathfrak{M}\) is a discrete set, finite
(Gromov compactness).

\[
\varphi(p) = 2 \cdot \# \mathfrak{M} (p, q, i, J, [u]) \quad \text{tw([u])}
\]

\[q \in \mathfrak{X}(\mathfrak{L}_0, L_1)\]

\([u]; \text{ind}(u) = 1\) ~ signed count if can orient \(\mathfrak{M}\).
Remarks 1) Given \( f \) we could have infinitely many \( L_i \) s.t.
\[ \text{Ind} = 1. \] But Gromov compactness says we must have only finitely many solutions
if we also impose \( w(L_i) \leq K \).

There are settings where we have a priori finiteness (e.g., of a priori energy estimates).

E.g., if \( M \) is exact symplectic & \( L_0, L_1 \) are exact Lagrangian, then \( w(L_i) \) is determined
by \( p \) & \( q \). Thus we can get rid of \( T \).

2) \( M \) can be oriented if we assume \( L_0, L_1 \)
are oriented & \( \text{spin} \).

3) **Transversality & compactness.**

- Ensure \( \mathcal{M} \) smooth by ensuring \( D_j \) onto
- Pick \( J \) generic, possibly \( t \)-dependent.
- If \( L_0 \) & \( L_1 \), then \( \square \)

Pick \( H: \mathcal{M} \times [0,1] \rightarrow \mathbb{R} \) Ham perturbation (generic)

\[
\frac{\partial u}{\partial s} + J(t, u(s,t)) \left( \frac{\partial u}{\partial t} - X_H(t, u(s,t)) \right) = 0
\]

\[
K((0,1), t) = \mathcal{K} : [0,1] \rightarrow \mathcal{M}
\]

If noncompact might have a large perturbation.
will bring in geometry from boundary.

Need to pick \( J \) & \( H \) together. See also Paul's book.

Good thinking.
\[ \Phi_{1,1}^{L_1,1}(x_1) = y(1) \in L_1 \]

\[ \nu(s, t) = \Phi_{1,1}^{L_1,1}(u(s, t)) \quad \infty \]

\[ \frac{\partial}{\partial s} + (\Phi, J)\left(\frac{\partial}{\partial t}\right) = 0 \]

Compactness, \( \tau^2 = 0 \)

In General Set Up

\[ \rightarrow \] Gromov compactness

A seq. of \( u_i \in M(p_i, q_i, J, \mathbb{C}^1) \) has a subseq. converging to a union of

- J hol (perturbed) strips \( \in M(p_i, q_i, \ldots) \)
- J hol discs w/ body on \( L_0 \) or \( L_1 \)
- J hol spheres

So a strip can converge to a chain of strips!

If spending more & more time here

Strip Breaking.

Bubbling:

Rescale at blow up of derv: see a disk or sphere.
get strip breaking, excluded if disk bubbling

\[ \sum_{\text{discs}} \omega = 0, \, 0 \]

disc bubbling, hard but a fact of life.

\[ \text{make } \mathcal{Z}^2 = 0 \]

sphere bubbling, annoying but (diam < 2 so shouldn't affect \( \mathcal{Z}^2 = 0 \)).

Stripping breaking: good, like in more many

\[ M(p, q, [u_1, J]) \] should be a 1D mod.

\[ \text{index } 2 \]

It admits a compactification by adding in being which consists of broken strips.

If \( \text{ind}(M(p, q, [u_1, J])) = 2 \)

\[ \bar{M}(p, q, [u_1, J]) = \bigoplus_{a \in \mathcal{L}(L_0, L_1)} M(p, q, [u_1, J], J_a) \times M(L_2, [u_1, J]) \]

\[ [u_1] = [u_1] + [u_2] \]

The index is additive provided we have transversality

\[ \text{ind}(M(p, q, [u_1, J])) = \text{ind}(M(u_1, J)) + \text{ind}(M(u_2, J)) \]

\[ 2 = 1 + 1 \]

\[ \check{v} = \gamma + 1 \]

coeff of \( \varphi \) in \( \mathcal{Z}^2 \) p counts \( p \to r \to q \).

total \# = 0.

both \( p \to q \)

\[ z(p) = (T^{a_1} - T^{a_2}) q \]

\[ z(q) = 0 \]

so \( M(\text{Rez}(\otimes, J), L_0) = 1 \).

\[ \mathcal{L}^2 \] if \( a_1 \neq a_2 \)

\[ 2^2 \neq 0. \text{ get disk bubbling shrinks and get } \]

\[ a_1 \]

\[ a_2 \]

\[ \text{good thinking} \]