$HF^*(L,L)$?

Local case: $L$ zero section $\subset T\mathbb{R}^2$

Pick $\varphi:\mathbb{R} \to \mathbb{R}$ Morse function (lift to $T\mathbb{R}^2 \to \mathbb{R}$)

$\varphi\circ (L) = \text{graph}(x \in H)$

$J(L,L,\varphi) = \text{Crit}(H)$

Clever choice of $J$: along zero-section, want

$J(\triangledown H) = \pm X_H$ (will identify $TL$ w/ $T^*L$ fibers, which is same as choosing a metric)

Floer's Thom: $HF^*(L,L) \cong H_{\text{Morse}}^*(H) = H^*(L)$ in $T^*L$.

Case of general $(M,W)$: understand Floer solos as $\varepsilon \to 0$

(Floer, Fukaya-Oh, Biran-Cornea, ...)

As $\varepsilon \to 0$, strips look like

Solutions converge to union of

- Gradient flow lines of $H$
- $T$-disk with boundary on $L$.

If no disks ($\text{cf. } \pi_2(H,L) = 0$), then $HF^*(L,L) \cong H^*(L)$.

If there are disks, might not even have $\partial^2 = 0$!

If $L$ monotone ($\mu(\text{disks}) = K \omega(\text{disks})$ for some $K > 0$)

or

-case of disks of $\mu < 2$ (except $\omega(\text{disks})$), then $\partial^2 = 0$.
- $\omega(\text{disks}) = \mu = 2$ disks are regular on $CF^*(L,L)$.
In such cases, can filter \( CF^* \) by Maslov index.

Get a spectral sequence (or) \( H^*(L) \Rightarrow HF^*(L, L) \)

(not \( \mathbb{Z} \)-graded, if non-trivial discs)

\[ \partial \beta = (T^a - T^a) \eta = 0 \]
\[ \partial \eta = 0 \]

in Morse theory:

\[ HF^*(L, L) = H^*(L) \]

\[ \beta \text{ in } C \]
\[ \partial \beta = (T^a - T^a) \eta = 0 \]
\[ \partial \eta = T^a \eta \]

\[ HF^* = 0 \]

\[ \eta \text{ equals in } CP^1 \]
\[ \partial \eta = (T^{A_1} - T^{A_2}) \rho = 0 \]
\[ HF^*(L, L) = H^*(S^1) \text{ if } A_1 = A_2 \]

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Products and Fukaya category

\[ L_0, L_1 \nRightarrow CF^*(L_0, L_1) = \bigoplus_{p \in \pi_1(L_0, L_1)} L_0 \cdot p \]

\[ \gamma \]
\[ \text{ counts index } -1 \text{ (perturbed) } J \text{-holo maps} \]

\[ R \times [0, 1] \xrightarrow{\gamma} L_1 \]

\[ R \times [0, 1] \quad \text{ (perturbed) } \quad \text{ maps} \]

\[ \gamma \]
Similarly,
\[ L_0, L_1, L_2 \sim CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \rightarrow CF^*(L_0, L_2) \]
(will be composition in Fukaya category)

Given \( p_1 \in \mathcal{X}(L_0, L_1), p_2 \in \mathcal{X}(L_1, L_2), q \in \mathcal{X}(L_0, L_2), \) have

\[
\begin{array}{c}
\[ L_0 \] \\
\hline
\begin{array}{c}
q \\
\hline
L_1 \\
\hline
L_2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\[ L_1 \] \\
\hline
\begin{array}{c}
p_2 \\
\hline
L_2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\[ L_2 \] \\
\hline
\begin{array}{c}
p_1 \\
\hline
L_0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\[ L_2 \] \\
\hline
\begin{array}{c}
p_2 \\
\hline
L_2
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\[ L_0 \] \\
\hline
\begin{array}{c}
p_1 \\
\hline
L_0
\end{array}
\end{array}
\]

(no non-trivial automorphisms of this domain)

\[ \mathcal{M}(p_1, p_2, q; [\mathcal{A}], J) = \{ u : \bigcirc \rightarrow M | \ \bar{\delta}_J u = 0, \ [\mathcal{A}] \text{ given,}\ ] u \rightarrow p_1, p_2, q \text{ at boundary arcs} \}
\]

Expected \( \dim = \text{index of linearized op.} = \text{ind}(\mathcal{A}J) = \)

\[ \deg(q) - \deg(p_1) - \deg(p_2) \]

when \[ \begin{cases} \text{Z-grad. (2C_1(TM)) = 0, Maslov classes (L_i) = 0} \\
\end{cases} \]

If linearized op. is onto, then \( \mathcal{M} \) is a manifold of this dim.

Def: Flux product \( = \) linear map given by

\[ p_2 \cdot p_1 = \sum_{q \in \mathcal{X}(L_0, L_2) \atop [\mathcal{A}] \text{ ind}(\mathcal{A}J) = 0} \# \mathcal{N}(p_1, p_2, q, [\mathcal{A}], J) T \rightarrow q \]

Proof: If no disk bubbling \( (\forall [\mathcal{A}], \pi_2(M, L_i) = 0 \ \forall i), \) then

\[ \partial (p_2 \cdot p_1) = \pm (\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1) \] (Leibniz rule)

and the induced product \( HF^*(L_1, L_2) \otimes HF^*(L_2, L_1) \rightarrow HF^*(L_0, L_2) \)

is independent of chosen \( J \) \& Ham perturb. and associative (unlike the product on \( CF^* \)).
Pf of Leibniz rule: considering index 1 $M(p_1, p_2, q, [u], J)$, have a 1-unfold which compactifies to a unifold w/ bubbling (we've assumed no sphere/disk bubbling).

Signed count of today pts of compactified index 1 $M$ is zero
$\Rightarrow \pm 2 (p_2 \cdot p_1) \pm (\partial p_2) \cdot p_1 \pm p_2 \cdot (\partial p_1) = 0$. 

Changing $J$ (as Han postulated) can study a continuation operation.

Let $J_1$ move left or right. At ends, get

There might also be solutions

These solutions can be used to write a homotopy between the product for $J_0$ & the product for $J_1$. 
Annunciative up to homotopy

Higher products: I sequence of operations

\[ \mu^k : CF^*(L_{k-1}, L_k) \otimes \cdots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_k) [2-k] \]

\[ \mu^1 = \partial, \quad \mu^2 = \text{product} \]

\[ \mu^k \text{ counts } J \text{- holomorphic curves} \]

\[ \left\{ \begin{array}{c}
\text{output } q \\
L_0 \\
L_1 \\
\vdots \\
L_k
\end{array} \right\} \xrightarrow{\eta} \left\{ \begin{array}{c}
p_k \\
L_0 \\
L_1 \\
\vdots \\
L_k
\end{array} \right\}, \quad \bar{\partial}_J \eta = 0 \]

\[ / \text{Aut}(D^2) \]

There's a (k-2) - dual family of such domains.

Expected dim \( \mathcal{M}(p_1, ..., p_k, q, [u], J) = \text{ind}([u]) + k - 2 \)

\[ \mu^k(p_k, - , p_1) = \sum_{q \in \Xi(L_0, L_k)} \# \mathcal{M}(p_1, ..., p_k, q, [u], J) \cdot q \]

\[ \text{Ind}([u]) = 2 - k \]

\[ \mu^k \text{ counts } J \text{- holomorphic curves} \]

\[ \text{dim } \mathcal{M}(p_1, ..., p_k, q, [u], J) = \text{ind }[u] + k - 2 \]

Prop: If \([w] \cdot \pi_2(M, L_i) = 0 \forall i\), then Ann - relations hold:

\[ \sum_{k=1}^{\infty} \sum_{j=0}^{k-2} (-1)^j \mu^{k-1-j} (p_j, - , \mu^j (p_{j+1}, ..., p_k)) = 0 \]

\[ \ast = j + \deg(p_j) + - + \deg(p_1) \]

\[ \begin{array}{c}
k=1: \quad \mu^1 (\mu^1 (p_1)) = 0 \\
k=2: \quad \text{Leibniz rule} \\
k=3: \quad \pm \mu^2 (\mu^2 (p_3, p_2), p_1) = \mu^2 (p_3, \mu^2 (p_2, p_1)) \pm \mu^3 (p_3, \mu^2 (p_2, p_1))
\end{array} \]

\[ \mu^2 \text{ associates, up to homotopy, given by } \mu^3 \]

\[ \mu^3 (p_3, p_2, \mu^1 (p_1)) \]
Pf of Acc-relation: In index $3-k$,

$M(p_{12}, p_k, q, \{H, J\})$ is a 1-fold, by compactification.

$\mathcal{M}_{0,k+1} = \left\{ \text{circle on } \mathbb{D}^2 \right\} / \text{Aut}(\mathbb{D}^2)$

Other models:

This space is contractible, by natural compactification to a polytope: the Hasegawa associahedron.

E.g.: $k=3$:

or

or

$tiss w/ inscribed horizontal rectangle$.

$k=4$: $\mathcal{M}_{0,5} =$

$\text{polytope}$
If \( \mathcal{A}(p_1, \ldots, p_k, 9, [u], J) \) is 1-dial, then generically have a pure family of domains:

- Codim 1 faces = pairs of disks
  (generically, avoid higher codim faces)

So, the boundary of a 1-dial \( \mathcal{A}(p_1, \ldots, p_k, 9, [u], J) \) is

(Trace energy escaping and breaking of domain)

Terms w/ \( \mu \)

Terms w/o \( \mu \)

**Note:** Naively, \( \mathbb{F} \) cyclic symmetry \( CF^*(L_0, L_1) \cong CF^{*+}(L_0, L_1) \).

For \( \langle \mu^k(p_k, \ldots, p_1), 9 \rangle \), \( \mathbb{F} \) cyclic symmetry, up to isomorphism.

Fukaya categories are Calabi-Yau Aoo-categories.

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**Hamiltonian perturbations**

We defined \( CF^*(L_0, L_1) \) using \( H_{20, L_1} \) & \( J_{20, L_1} \).

For product:

- \( H_{12}, J_{12} \) & \( L_2 \)
- \( H_{02}, J_{02} \) & \( L_0 \)
- \( H_{20}, J_{20} \) & \( L_0 \)
Choose strip-like ends: charts $y = s + it$ near punctures.

Eq: $\frac{\partial a}{\partial s} + J^t \left( \frac{\partial a}{\partial t} - X_{4t} \right) = 0$ near ends.

This is the Floer eq w/ exact perturbations near ends.

That's it for strip-breaking.

What about domain-breaking?

\[ \begin{array}{c}
H_0 & \sim & H_0 \\
H_1 & \sim & H_1 \\
H_2 & \sim & H_2
\end{array} \]

H must vary differently w/ different domains!

Thus (Seidel): 3 inductive procedure for constructing consistent families of $(H,J)$.

The set of choices at each step is constructible (see Seidel's book).

This uses fact that association is constructible.

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**Fukaya category**

$(M,\omega)$ with or w/ reasonable behavior at infinity.

**Objs** = compact Lagrangian submanifolds, unobstructed (no holom. discs), + spin structures (for char $\neq 2$),

(+ grading if $\chi_1(M) = 0$ and ward $\mathbb{Z}$-grading),

+ local system ("unitary" flat bundle $L$).