$F_\omega(CP^1) =$ every weakly nontrivial Lagr. of $\mu^0 = \omega, 1^\exists$

is an honest $\mathbb{H}$-category for all well-$k$.

$\mu^0 = W(\zeta). \text{id}$

$W(\zeta) = 3 + \frac{t^A}{3}$

Have an object of $F_\omega(\zeta)$, but get $HF(\zeta, \zeta) = 0$ unless $\zeta$ is an equator: $\alpha = \frac{A}{2}$. (otherwise, $\zeta$ is displacable!)

Also must have $\text{id} \pm 1$.

So, non-trivial objs. ($HF(\zeta, \zeta) \neq 0$) are at $\zeta = \pm 2 T^{\frac{A}{2}}$.

$\Rightarrow W = 3 + \frac{t^A}{3} = \pm 2 T^{\frac{A}{2}}.$

These are the critical $\mu^0$ of $W$.

**Mirror:** $MF_w(X^w, W)$ matrix factorizations.

**Assume:** $X^w = \text{Spec } R$ affine.

$R^{\otimes k} \xrightarrow{\phi} R^{\otimes k}, \quad \begin{bmatrix} \phi \end{bmatrix} = (W - w). \text{id}$ matrix factorization

$MF(\mathcal{K}^w, 3 + \frac{t^A}{3} - w)$ non-trivial iff $W = \pm 2 T^{\frac{A}{2}}$.

$\mathcal{K}[3 \pm 1^\text{st}] \xrightarrow{f.g} \mathcal{K}[3 \pm 1^\text{st}] \quad f.g = W - w$

$3 + \frac{t^A}{3} \pm 2 T^{\frac{A}{2}} = (3 - T^{\frac{A}{2}}) (1 - \frac{t^{A/2}}{3})$

This non-trivial matrix fact is mirror to equal $W/\text{hol} = \pm 1$.

**Order:** $MF(W - w) \simeq D^b \text{Sing } (W - w) := D^b \text{Coh } (W^w(W)) / \text{Perf}$
Recap: $X$ Kähler $\Rightarrow$ D linear (reduced, normal crossing)

\[ \text{Id} \cdot I \cdot D = -K_X \]

$X^0 = X \setminus D$ is open CY

**Candidate mirrors:**

$X'$ mirror to $X^0$: "moduli space" of $T^n$-objects of $F(X^0)$

(eg: log tori w/ rank 1 local syst.)

- Family Floer theory (Abouzaid, Furuta) gives a more robust approach (eventually). So far: log torus fibr w/ sig. singularities,
  Furuta: simple sig. singularities in dim 4.

- SYZ: Input = log $T^n$-fibr on $X^0$, w/a logr. section.

$W \in \mathcal{O}(X')$ measures how logs become nearly undisturbed objects in $F(X)$: $\mu^0 = W \cdot \text{id}$.

($W$ is a weighted count of dual. discs w/ $[\mu \cdot D = 1]$)

**Ex:** $(\mathbb{C}^*)^n \leftrightarrow (\mathbb{C}^*)^n$ or $(\mathbb{R}^*)^n$

If $X$ toric Fano, $D$ = toric divisors, then

$X \leftrightarrow X^\vee = (\mathbb{C}^*)^n$, $W$ = Laurent poly whose terms are given by facets of polytope

rings of fans
Ex: \( \mathbb{CP}^1 \leftrightarrow \mathbb{C}^* \), \( W = z + \frac{T^A}{z} \)
\( \mathbb{CP}^2 \leftrightarrow (\mathbb{C}^*)^2 \), \( W = x + y + \frac{T^A}{xy} \)

\( D = \Delta \)

Ex (non-trivic):

\( C^2 \setminus \{uv=13\} \leftrightarrow C^2 \setminus \{uv=15\} \)

\( (\mathbb{CP}^2, D = \Delta) \leftrightarrow C^2 \setminus \{uv=13\} \), \( W = u + \frac{T^A}{uv} \)

This mirror to \( \mathbb{CP}^2 \) is equivalent to a non-trivic mirror above!

Non-trivic mirror

\( (\mathbb{C}^*)^2 \)

\( \text{cut vals} = 3 \pm \sqrt{A \cdot T^{A/3}} \)

\( \text{cut pts} \quad x = y = \sqrt[3]{A \cdot T^{A/3}} \)

Uni-trivic mirror

\( C^2 \setminus \{uv=13\} \)

Each fiber has one more point!

by the two geometries

The relation can be seen as arising from

\( (\mathbb{C}^*)^2 \) open dense \( C^2 \setminus \{uv=13\} \)

\( (x, y) \mapsto (u, v) = (x + y, x^{-1}) \)

The image is a complete section

and further compactify to

\( (\mathbb{CP}^2, D = \text{smooth cubic}) \), fibers = \( \mathbb{C} \) elliptic curves

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General Principle: Fibers are mirror to \( D \). \( \otimes \) mirror to \( D = T^2 \).

In this case, \( \circ \) mirror to \( D = \Delta \); \( \odot \) mirror to partially smoothed \( D = \mathbb{C}^2 \)
There are a lot of charts \((\mathbb{C}^*)^2\) with potentials given by Laurent polynomials.

The mirror of \(\mathbb{CP}^2\) is smooth. \(D\) has a "cluster structure", i.e., a collection of distinguished \((\mathbb{C}^*)^2\) charts, related by birational tranfs, st expression of \(W\) is a Laurent poly in all these charts.

These charts \(\leftrightarrow\) worth \(\mathbb{CP}^2\) in \(\mathbb{CP}^2\) [Vieira]

\(\leftrightarrow\) tonic degenerations of \(\mathbb{CP}^2\) to \(\mathbb{CP}^2(a^2, b^2, c^2)\), indexed by \((a, b, c)\) for Newton triples \((a, b, c) : a^2 + b^2 + c^2 = 3abc\) [Hacking-Prokhorov]

Long: These are all monomial tori in \(\mathbb{CP}^2\). Not clear for other fibr... (Guth + etc. + akin...: get classification of Fano 3-folds by studying )

"reasonable" Laurent polynomials and that transform into other such Laurent polynomials under mutation)

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**HMS #1**

\(F(X)\) is \(A_\infty\) deform (curved, \(\mu^0\), not 2-graded) of \(F(X^0)\), described by \(\alpha_0 \in HH^0(F(X^0)) \leftrightarrow O(X^0) \Rightarrow W\).

\(F(X)_\lambda\) (uncurved) \(A_\infty\) categ,

\(\text{Objs} = \text{heavily obstructed} \) \(\text{Lags in} \ F(X)\), s.t \(\mu^0 = \lambda\text{id}\).

**HMS**:

\[D^\pi F(X)_\lambda = D^\pi \text{sing} (W^{-1}(\lambda)) \cong \text{MF}(W-\lambda) \quad (= 0 \text{ if } \lambda \notin \text{Int}(W))\]
Ex: \( CP^1 \)  
\[ A_2 = \frac{\mathbb{C}}{\mathbb{Z}_2} \]
\[ \text{hol} = \pm 1 \]
\[ \lambda = \pm 2 T^{A/2} \]

\[ W - \Lambda = 3 + \frac{I_2}{3} + 2 T^{A/2} \]
\[ = (3 + T^{A/2})(1 + 3^{-1} T^{A/2}) \]

Ex: \( CP^2 \)  
\[ \text{Clifford} = \{ (x,y,z) \mid 1x1 = 1y1 = 1z1 \} \]

\[ \Delta \]

w/ 3 local systems, s.t. \( HF(T,T) = H^*(T^2) \) as vec space  
(r.f.g: Clifford algebra = Mat_{2x2})  
(often fibres are degenerate, i.e. parallelizable)

\[ \lambda = 3 \nu^2 T^{A/3} \]

The 3 crit pts of \( W = x + y + \frac{I_2}{xy} \) give rise to \( HF \)’s.

Exercise: \( D_{\text{sing}}(\{xy = 0 \} \subset \mathbb{C}^2) \) vs \( MF(\mathbb{C}^2, xy) \)

\[ \text{Col/Pref} \]

L sheaves given by finite complexes of \( f \)-flat vector bundles.

No finite reduction of \( O_A \) by vector bundles: for example

\[ 0 \to O^x \to O^y \to O_A \to 0 \]

\[ \text{reduction initially } 2 \text{-periodic} \]

\[ xy = 0 \]

\[ 0_A \] is Perf

\[ MF \left[ O \frac{\mathcal{O}_x}{x} 0 \right] \]

\[ A = x \text{-axis, } y = 0 \]
Have SES

\[ 0 \rightarrow O_B \rightarrow O \rightarrow O_A \rightarrow 0 \]
\[ 0 \rightarrow O_A \oplus O_B \rightarrow O \rightarrow O_{\mathfrak{p}=0} \rightarrow 0 \]

\[ \Rightarrow \text{ in } D^b_{\Sigma_{\gamma}} \text{, } O_B \cong O_A \langle 1 \rangle \text{, } O_{\mathfrak{p}=0} \cong O_A \oplus O_B \langle 1 \rangle \]

\[ \text{End}(O_{\mathfrak{p}=0}) \cong \text{Mat}_{2 \times 2} \cong H^2(T^2) \text{ deformed} \]

\[ \text{Ex} : \text{ toric Fanov } : A^{000} \]

\[ F(X) \] split - generated by large tori (w/ local syst. + boundary chains)

For genericayment class \([\omega] \), \( F(X) \) consists of tori

\[ \Rightarrow \text{ mirror } W \text{ has isolated non-deg cut pts (as before)} \]

For specific \([\omega] \), situation might be richer:

\[ \mathbb{C}P^1 \times \mathbb{C}P^1 \]
\[ (T = S^1 \times S^1 \pm \pm) \]

\[ \text{if } A_1 = A_2 ! \]

\[ U = x + y + \frac{T^{A_1}}{x} + \frac{T^{A_2}}{y} \]

\[ \text{if } A_1 = A_2 ! \]
Other side of HMS: \( \text{D}^b \text{Co}(X) = \text{FS}(X^v, W) \)

FS: \( L \subset X^v \)

goes to right - i.e.

\[ \text{X} \quad \text{W} \to \infty \]

Jones: Large flow then \( W \) Not hot at all in base \( \uparrow \to \infty \)

(if \( W \) is not proper, wrap in fiber directions

- so, wrap in fibers and rotate a little in base)

Ex: \( (\mathbb{C}^\times, z + \frac{t^a}{z}) \)

\[ \begin{array}{c}
\text{End} (L_1) = 1K \\
\text{Hom} (L_1, L_0) = 0, \quad \text{Hom} (L_0, L_1) = 1K^2 \\
\text{Seidel: the two twists generate FS}
\end{array} \]

HMS can be proved in this way:

- Del Pezzo surfaces \([A - Karzanov - Orlov]\)
- Toric varieties \([Abouzaid;FLT2]\)
Have several functors

1) **One-term mapping**: \( \text{FS}(X^v, W) \to \phi_! \)

This comes with a natural transformation \( \phi_! \to \text{id} \)

\[ \downarrow \text{mirror} \quad \phi_! \]

\[ \text{D}^b \text{Coh}(X) \to \text{D}^b \text{Coh}(X^v) \]

\[ (\text{Seidel}) \]

with a natural transformation \( \text{O}(-D) \to \text{id} \)

\[ \times \text{Spec}_D \]

This gives an autoequivalence of \( X^v \), since we can undo it, twisting the other way.

2) **Acceleration functor**: \( \text{FS}(X^v, W) \to W(X^v) \)

\[ \uparrow \text{mirror} \quad \phi_! \to \text{id} \]

\[ \text{D}^b \text{Coh}(X) \to \text{D}^b \text{Coh}(X \times \text{Spec}_D) \]

\[ \text{(localize wrt \( \times \text{Spec}_D \))} \]

if fiber not split

3) **Restriction to fiber**

\[ l = l_1 \text{fiber} \quad \text{get} \quad \text{FS}(X^v, W) \to \text{F(fiber)} \]

\[ \text{mirror} \quad \phi_! \to \text{id} \]

\[ \text{D}^b \text{Coh}(X) \to \text{D}^b \text{Coh}(D) \]

\[ \text{(derived)} \]

\[ \text{adjoint functors} \]

\[ \text{F(fiber)} \to \text{FS}(X^v, W) \]