$GL(n, \mathbb{Z})$ := group of $n \times n$ invertible matrices

$n=2$

$a := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ b := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Claim: $a \& b$ generate a free subgroup of $GL(2, \mathbb{Z})$!

Idea: Let's study the action of $a \& b$ on $\mathbb{Z}^2$.

$GL(2, \mathbb{Z}) \ast \mathbb{Z}^2$ by matrix-vector multiplication.

eg. $a \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{(for all } k \in \mathbb{Z})$

#1. $a^2 = (\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix})^2 = \begin{pmatrix} 4 & 4 \\ 0 & 1 \end{pmatrix} \ldots, \ a^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}.$

#2. when $|x| < |y|$, $a^k \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2ky \\ y \end{pmatrix}$

and $|x + 2ky| \geq \frac{2ky}{|x|} \Rightarrow |x| > |y|$.
※ Things in the red zone get sent into things in the black zone, and they stay there too!

#3. b does the opposite, sending things in the black zone to things in the red zone.

#4. How does this help us? PING-PONG!

Recall the definition of a group action

※ \( I_2 \cdot v = v \) for all \( v \in \mathbb{Z}^2 \).

Hence, if some word \( w \) in \( a \) & \( b \) is equal to \( I_2 \), then that word fixes everything!
Choose any $v$ in the black zone and see that $(a^3 b^2 a) \cdot v \notin v$ in the red zone.

That is, $(a^3 b^2 a) \cdot v \neq v$.

So $a^3 b^2 a \neq 1_2$, i.e., $a^3 b^2 a$ is not a relation.

In fact, $\langle a, b \rangle$ has no relations.

Lemma ("Ping-Pong for two players")

- $G$ generated by $a$ and $b$.
- $G \cap X$ nonempty.
- Have $X_a, X_b \subseteq X$: $a^k(X_b) \subseteq X_a$ & $b^k(X_a) \subseteq X_b$.

$\Rightarrow G_1 \cong F_2$ free group of rank 2.
$\mathbb{C}P := $ space of 1-dimensional subspaces of $\mathbb{C}^2$ with antipodal points identified. 

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$\cong \text{ "Riemann sphere" }$

$\uparrow \text{ This is a complex manifold!}$

Möbius Transformations

$Z \mapsto \frac{aZ+b}{cZ+d}$ where $ad-bc \neq 0$. 

$\left( -\frac{d}{c} \mapsto \infty, \infty \mapsto \frac{a}{c} \right)$

$\ast \left( Z \mapsto \frac{a'Z+b'}{c'Z+d'} \right) \circ \left( Z \mapsto \frac{aZ+b}{cZ+d} \right) = \left( Z \mapsto \frac{(aa'+bc)b+(a'b+c'd)(c'a+d'c)a+bc'a'}{(c'a+d'c)a+(bc')d} \right)$
Have a natural identification of Möbius transformations with 2x2 matrices over \( \mathbb{C} \):

\[
(z \mapsto \frac{az + b}{cz + d}) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\(ad - bc \neq 0 \iff \leftrightarrow \in \text{GL}(2, \mathbb{C})\).

Almost...

Not well-defined because

\[
\frac{az + b}{cz + d} = \frac{(2a)z + (2b)}{(2c)z + (2d)}
\]

Solution:

\[
\frac{\text{GL}(2, \mathbb{C})}{\langle \text{I}_2 \rangle} \cong \frac{\text{PGL}(2, \mathbb{C})}{\text{PSL}(2, \mathbb{C})}
\]

Tchotthy groups over disks

\( 2g \) disjoint circles (with disjoint interiors) \( A_1, B_1, \ldots, A_g, B_g \).
Möbius transformation so that \( T_i(A \cdot c) \subseteq B \).

Since the set of all Möbius transformations is a group, the \( T_i \) generate a subgroup of this group, and under the identification with \( \text{PSL}(2, \mathbb{C}) \), a subgroup of \( \text{PSL}(2, \mathbb{C}) \).

Such a subgroup is called a "classical Schottky group."

Theorem (Maskit 1967)

Theorem [1], and the planarity theorem [3]. A finitely generated Kleinian group \( G \) is a Schottky group if and only if \( G \) is free, and every element of \( G \) other than the identity is loxodromic (hyperbolic transformations are included among the loxodromic).
Baby version: (classical) Schottky groups are free!

Proof. Apply the Ping-Pong Lemma. □